Introduction to 3+1 Formalism

2021 Summer School on Numerical Relativity and Gravitational Waves

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1 Preliminaries

:

$c = 1$: natural unit
$8\pi G = 1$
$a, b, c, \cdots$: abstract indices [3]
$\mu, \nu, \cdots$: 0,1,2,3
$i, j, k, \cdots$: 1,2,3
Lower case letter: spacetime tensors
Upper case letter: spatial tensors (exceptions: induced metric γ, induced Levi-Civita connection ϵ)
Letter convention: [2]
Approach: [1]

2 Geometrical Quantities

2.1 Given Structure

Let us consider a globally hyperbolic spacetime $(\mathcal{M}, g)$ with a time function t whose level surface Σ_t is spacelike Cauchy surface (see Chapter 10 in [3]). We assume that t increases along any future directed timelike path. For any timelike vector v,

$$\begin{cases} v(t) > 0 & : v \text{ is future directed} \\ v(t) < 0 & : v \text{ is past directed} \end{cases}. \tag{1}$$

2.2 Normal Vector

The graident dt is timelike:

$$g^{ab} (dt)_a (dt)_b < 0. \tag{2}$$

We define lapse N, which is normalization factor for dt, as

$$N \equiv \frac{1}{\sqrt{-g^{ab} (dt)_a (dt)_b}}. \tag{3}$$

The normal vector n is defined as a vector that is metric dual to dt and unit given by

$$n^a \equiv -N g^{ab} (dt)_b \,. \tag{4}$$

The minus sign was chosen so that n is future directed:

$$n(t) = n^a (dt)_a \tag{5}$$
$$= N^{-1} \tag{6}$$
$$> 0. \tag{7}$$

2.3 Projections

For an arbitrary vector v, projection onto n is given by

$$v_\parallel^a \equiv -n^a n_b v^b. \tag{8}$$

The minus sign is required because n is timelike. It is justified by

$$n_a v_{\parallel}^a = -n_a n^a n_b v^b \tag{9}$$

$$= n_a v^a. \tag{10}$$

A vector v is temporal if and only if $v^a = -n^a n_b v^b$.

Let us consider orthogonal projection of v onto Σ_t given by

$$v_{\perp}^a = v^a - v_{\parallel}^a \tag{11}$$

$$= v^a + n^a n_b v^b \tag{12}$$

$$= \left(\delta^a{}_b + n^a n_b\right) v^b. \tag{13}$$

It suggests defining an orthogonal projector given by

$$\gamma^a{}_b = \delta^a{}_b + n^a n_b. \tag{14}$$

The orthogonal projector is orthogonal to n, has trace of 3, and is idempotent:

$$\gamma^a{}_b n_a = 0, \tag{15}$$

$$\gamma^a{}_b n^b = 0, \tag{16}$$

$$\gamma^a{}_a = 3, \tag{17}$$

$$\gamma^a{}_c \gamma^c{}_b = \gamma^a{}_b. \tag{18}$$

A vector is spatial if and only if $v^a = \gamma^a{}_b v^b$.

2.4 Orthogonal Decomposition

A vector v is decomposed into

$$v^a = \mathfrak{A} n^a + \mathfrak{B}^a, \tag{19}$$

where $\mathfrak{A}$ is a scalar of temporal component and $\mathfrak{B}$ is a spatial vector of spatial component. Components are determined by

$$\mathfrak{A} = -n_a v^a, \tag{20}$$

$$\mathfrak{B}^a = \gamma^a{}_b v^b. \tag{21}$$

A form ω is decomposed into

$$\omega_a = \mathfrak{C} n_a + \mathfrak{D}_a, \tag{22}$$

where

$$\mathfrak{C} = -n^a \omega_a, \tag{23}$$

$$\mathfrak{D}_a = \gamma^b{}_a \omega_b. \tag{24}$$

A rank (0,2) tensor is decomposed into

$$x_{ab} = n_a n_b \mathfrak{A} + n_a \mathfrak{B}_a + \mathfrak{C}_a n_b + \mathfrak{D}_{ab}, \tag{25}$$

where

$$\mathfrak{A} = n^a n^b x_{ab}, \tag{26}$$

$$\mathfrak{B}_a = -n^c \gamma^d{}_a x_{cd}, \tag{27}$$

$$\mathfrak{C}_a = -\gamma^c{}_a n^d x_{cd}, \tag{28}$$

$$\mathfrak{D}_{ab} = \gamma^c{}_a \gamma^d{}_b x_{cd}. \tag{29}$$

2.5 Spatial Metric

Observing components of the metric by

$$n^a n^b g_{ab} = -1, \tag{30}$$

$$-n^c \gamma^d{}_a g_{cd} = 0, \tag{31}$$

$$-\gamma^c{}_a n^d g_{cd} = 0, \tag{32}$$

$$\gamma^c{}_a \gamma^d{}_b g_{cd} = \gamma_{da} \gamma^d{}_b \tag{33}$$

$$= \gamma_{ad} \gamma^d{}_b \tag{34}$$

$$= \gamma^c{}_d \gamma^d{}_b g_{ca} \tag{35}$$

$$= \gamma^c{}_b g_{ca} \tag{36}$$

$$= \gamma_{ab}, \tag{37}$$

we get

$$g_{ab} = -n_a n_b + \gamma_{ab}. \tag{38}$$

We can interpret γ_{ab} as a metric in Σ_t because, for spatial vectors X and Y,

$$\gamma_{ab} X^a Y^b = \gamma^c{}_a \gamma^d{}_b g_{cd} X^a Y^b \tag{39}$$
$$= g_{ab} X^a Y^b. \tag{40}$$

2.6 Spatial Levi-Civita Tensor

Let us define the induced Levi-Civita tensor by

$$\epsilon_{abc} \equiv n^d \varepsilon_{dabc}, \tag{41}$$

where ε_{dabc} is the spacetime Levi-Civita tensor. It is spatial becuase

$$\epsilon_{abc} n^c = n^d \varepsilon_{dabc} n^c \tag{42}$$
$$= 0. \tag{43}$$

It has properties of

$$\epsilon^{abc}\epsilon_{def} = 6\gamma^{[a}{}_d \gamma^b{}_e \gamma^{c]}{}_f. \tag{44}$$
$$\epsilon^{abe}\epsilon_{cde} = 2\gamma^{[a}{}_c \gamma^{b]}{}_d, \tag{45}$$
$$\epsilon^{acd}\epsilon_{bcd} = 2\gamma^a{}_b, \tag{46}$$
$$\epsilon^{abc}\epsilon_{abc} = 6, \tag{47}$$

2.7 Irreducible Decomposition of Spatial Rank (0,2) Tensors

A spatial rank (0,2) tensor X is decomposed into symmetric and antisymmetric part:

$$X_{ab} = X_{(ab)} + X_{[ab]}. \tag{48}$$

The symmetric part is decomposed into trace and traceless part:

$$X_{(ab)} = \frac{1}{3}\gamma_{ab} X + \breve{X}_{ab}, \tag{49}$$

where

$$X = \gamma^{ab} X_{ab}, \tag{50}$$
$$\breve{X}_{ab} = \left(\gamma^{(c}{}_a \gamma^{d)}{}_b - \frac{1}{3}\gamma_{ab}\gamma^{cd}\right) X_{cd}. \tag{51}$$

The antisymmetric part is expressed by the Hodge dual vector as

$$X_{[ab]} = \epsilon^c{}_{ab} X_c, \tag{52}$$

where

$$X_a = \frac{1}{2}\epsilon^{bc}{}_a X_{bc}. \tag{53}$$

As a result, a spatial rank (0,2) tensor is decomposed into

$$X_{ab} = \frac{1}{3}\gamma_{ab} X + \breve{X}_{ab} + \epsilon^c{}_{ab} X_c. \tag{54}$$

The number of components for quantities are summarized in

- X: 1
- $\breve{X}_{ab}$: 5
- X_a: 3
- Total: 9

2.8 Orthogonal Decomposition of Symmetric and Antisymmetric Tensors

A symmetric tensor x is decomposed into

$$x_{ab} = n_a n_b \mathfrak{A} + 2n_{(a}\mathfrak{B}_{b)} + \frac{1}{3}\gamma_{ab}\mathfrak{C} + \breve{\mathfrak{C}}_{ab}, \tag{55}$$

where

$$\mathfrak{A} = n^a n^b x_{ab}, \tag{56}$$

$$\mathfrak{B}_a = -n^c \gamma^d{}_a x_{cd}, \tag{57}$$

$$\mathfrak{C} = \gamma^{ab} x_{ab}, \tag{58}$$

$$\breve{\mathfrak{C}}_{ab} = \left(\gamma^{(c}{}_a\gamma^{d)}{}_b - \frac{1}{3}\gamma_{ab}\gamma^{cd}\right) x_{cd}. \tag{59}$$

The number of components for quantities are summarized in

- $\mathfrak{A}$: 1
- $\mathfrak{B}_a$: 3
- $\mathfrak{C}$: 1
- $\breve{\mathfrak{C}}_{ab}$: 5
- Total: 10

A antisymmetric tensor y is decomposed into

$$y_{ab} = 2n_{[a}\mathfrak{D}_{b]} + \epsilon^c{}_{ab}\mathfrak{E}_c, \tag{60}$$

where

$$\mathfrak{D}_a = y_{ab} n^b, \tag{61}$$

$$\mathfrak{E}_a = \frac{1}{2}\epsilon^{bc}{}_a y_{bc}. \tag{62}$$

The number of components for quantities are summarized in

- $\mathfrak{D}_a$: 3
- $\mathfrak{E}_a$: 3
- Total: 6

2.8.1 Example: Stress-energy Tensor

The stress-energy tensor t is decomposed into

$$t_{ab} = n_a n_b \rho + n_a P_b + P_a n_b + \frac{1}{3}S + \breve{S}_{ab}, \tag{63}$$

where E is the energy density, P is the momentum density, and $S_{ab} = \frac{1}{3}\gamma_{ab}S + \breve{S}_{ab}$ is the stress. Components are given by

$$\rho = n^a n^b t_{ab}, \tag{64}$$

$$P_a = -n^c \gamma^d{}_a t_{cd}, \tag{65}$$

$$S = \gamma^{ab} t_{ab}, \tag{66}$$

$$\breve{S}_{ab} = \left(\gamma^{(c}{}_a\gamma^{d)}{}_b - \frac{1}{3}\gamma_{ab}\gamma^{cd}\right) t_{cd}. \tag{67}$$

2.8.2 Example: Field Strength Tensor

The field strength tensor f is decomposed into

$$f_{ab} = 2n_{[a}E_{b]} + \epsilon^c{}_{ab}B_c, \tag{68}$$

where E is the electric field and B is the magnetic field. Components are given by

$$E_a = f_{ab} n^b, \tag{69}$$

$$B_a = \frac{1}{2}\epsilon^{bc}{}_a f_{bc}. \tag{70}$$

2.9 Ricci Tensor and Scalar

From the Einstein equation, we get the Ricci tensor r as

$$r_{ab} = t_{ab} - \frac{1}{2} g_{ab} t \tag{71}$$

$$= n_a n_b \rho + n_a P_b + P_a n_b + S_{ab} - \frac{1}{2} \left(-n_a n_b + \gamma_{ab}\right)(S - \rho) \tag{72}$$

$$= \frac{1}{2}(S + \rho) n_a n_b + n_a P_b + P_a n_b + \frac{1}{6} \gamma_{ab} (3\rho - S) + \check{S}_{ab}. \tag{73}$$

The Ricci scalar becomes

$$r = -\frac{1}{2}(S + \rho) + \frac{1}{2}(3\rho - S). \tag{74}$$

$$= \rho - S \tag{75}$$

The traceless Ricci tensor is given by

$$\bar{r}_{ab} = r_{ab} - \frac{1}{4} g_{ab} r \tag{76}$$

$$= \frac{1}{2} n_a n_b (S + \rho) + n_a P_b + P_a n_b + \frac{1}{6} \gamma_{ab} (3\rho - S) + \check{S}_{ab} - \frac{1}{4}\left(-n_a n_b + \gamma_{ab}\right)(\rho - S) \tag{77}$$

$$= \frac{1}{4} n_a n_b (3\rho + S) + n_a P_b + P_a n_b + \frac{1}{12} \gamma_{ab} (3\rho + S) + \check{S}_{ab}. \tag{78}$$

2.10 Ricci Decomposition of the Riemann Tensor

Consider rank (0,4) tensor x. It is decomposed into the form given by

$$x_{abcd} = \mathfrak{a} g_{ab} g_{cd} + \mathfrak{b} g_{ac} g_{bd} + \mathfrak{c} g_{ad} g_{bc} + g_{ab} \mathfrak{d}_{cd} + g_{ac} \mathfrak{e}_{bd} + g_{ad} \mathfrak{f}_{bc} + \mathfrak{g}_{abcd}, \tag{79}$$

where $\mathfrak{d}$, $\mathfrak{e}$, $\mathfrak{f}$, and $\mathfrak{g}$ are traceless tensors such that their self contractions of any pair of indices vanish. Let us antisymmetrize pairs of ab and cd:

$$x^{[ab]}{}_{[cd]} = (\mathfrak{b} - \mathfrak{c})\, \delta^{[a}{}_{[c} \delta^{b]}{}_{d]} + \delta^{[a}{}_{[c} \left(\mathfrak{e}^{b]}{}_{d]} - \mathfrak{f}^{b]}{}_{d]} \right) + \mathfrak{g}^{[ab]}{}_{[cd]}. \tag{80}$$

Observing this, we conclude that Riemann tensor is decomposed into

$$r^{ab}{}_{cd} = \frac{1}{6} r \delta^{[a}{}_{[c} \delta^{b]}{}_{d]} + 2\delta^{[a}{}_{[c} \bar{r}^{b]}{}_{d]} + c^{ab}{}_{cd}, \tag{81}$$

where c is the Weyl tensor such that its contractions of any pair of indices vanish.

2.11 Orthogonal Decomposition of the Riemann Tensor

2.11.1 Rank (2,2) Tensor of Riemann Tensor Type

A rank (2,2) tensor x antisymmetric to covariant and contravariant indices resepectively, satisfying

$$x^{(ab)}{}_{cd} = 0, \tag{82}$$

$$x^{ab}{}_{(cd)} = 0, \tag{83}$$

is decomposed into

$$x^{ab}{}_{cd} = 2n^{[a} \left(2n_{[c} \mathfrak{A}^{b]}{}_{d]} + \epsilon^{|f|}{}_{cd} \mathfrak{B}^{b]}{}_{f} \right) + \epsilon^{ab}{}_{e} \left(2n_{[c} \mathfrak{C}^{e}{}_{d]} + \epsilon^{f}{}_{cd} \mathfrak{D}^{e}{}_{f} \right) \tag{84}$$

$$= 4n^{[a} n_{[c} \mathfrak{A}^{b]}{}_{d]} + 2n^{[a} \epsilon^{|f|}{}_{cd} \mathfrak{B}^{b]}{}_{f} + 2\epsilon^{ab}{}_{e} n_{[c} \mathfrak{C}^{e}{}_{d]} + \epsilon^{ab}{}_{e} \epsilon^{f}{}_{cd} \mathfrak{D}^{e}{}_{f}, \tag{85}$$

where

$$\mathfrak{A}^{a}{}_{b} = n_c n^d x^{ac}{}_{bd}, \tag{86}$$

$$\mathfrak{B}^{a}{}_{b} = \frac{1}{2} n_c \epsilon^{de}{}_{b} x^{ac}{}_{de}, \tag{87}$$

$$\mathfrak{C}^{a}{}_{b} = \frac{1}{2} \epsilon^{a}{}_{de} n^c x^{de}{}_{bc}, \tag{88}$$

$$\mathfrak{D}^{a}{}_{b} = \frac{1}{4} \epsilon^{a}{}_{cd} \epsilon^{ef}{}_{b} x^{cd}{}_{ef}. \tag{89}$$

The number of components for quantities are summarized in

- $\mathfrak{A}^a{}_b$: 9
- $\mathfrak{B}^a{}_b$: 9
- $\mathfrak{C}^a{}_b$: 9
- $\mathfrak{D}^a{}_b$: 9
- Total: 36

2.11.2 First Term

The first term in eq. (81) is decomposed into

$$\begin{aligned}\frac{1}{6}r\delta^{[a}{}_{[c}\delta^{b]}{}_{d]} &= -\frac{1}{6}rn^{[a}n_{[c}\delta^{b]}{}_{d]} + \frac{1}{6}r\gamma^{[a}{}_{[c}\delta^{b]}{}_{d]} && (90)\\ &= -\frac{1}{6}rn^{[a}n_{[c}\gamma^{b]}{}_{d]} - \frac{1}{6}r\gamma^{[a}{}_{[c}n^{b]}n_{d]} + \frac{1}{6}r\gamma^{[a}{}_{[c}\gamma^{b]}{}_{d]}. && (91)\end{aligned}$$

Note that

$$\begin{aligned}\gamma^{[a}{}_{[c}\gamma^{b]}{}_{d]} &= \frac{1}{2}\left(\gamma^{[a}{}_{c}\gamma^{b]}{}_{d} - \gamma^{[a}{}_{d}\gamma^{b]}{}_{c}\right) && (92)\\ &= \frac{1}{2}\left(\gamma^{[a}{}_{c}\gamma^{b]}{}_{d} - \gamma^{[b}{}_{c}\gamma^{a]}{}_{d}\right) && (93)\\ &= \frac{1}{2}\left(\gamma^{[a}{}_{c}\gamma^{b]}{}_{d} + \gamma^{[a}{}_{c}\gamma^{b]}{}_{d}\right) && (94)\\ &= \gamma^{[a}{}_{c}\gamma^{b]}{}_{d} && (95)\\ &= \frac{1}{2}\epsilon^{ab}{}_{e}\epsilon^{e}{}_{cd}. && (96)\end{aligned}$$

Then,

$$\begin{aligned}\frac{1}{6}r\delta^{[a}{}_{[c}\delta^{b]}{}_{d]} &= -\frac{1}{3}rn^{[a}n_{[c}\gamma^{b]}{}_{d]} + \frac{1}{12}r\epsilon^{ab}{}_{e}\epsilon^{e}{}_{cd} && (97)\\ &= 4n^{[a}n_{[c}\left\{-\frac{1}{12}\gamma^{b]}{}_{d]}\left(\rho - S\right)\right\} + \epsilon^{ab}{}_{e}\epsilon^{f}{}_{cd}\left\{\frac{1}{12}\gamma^{e}{}_{f}\left(\rho - S\right)\right\}. && (98)\end{aligned}$$

2.11.3 Second Term

$$\begin{aligned}2\delta^{[a}{}_{[c}\bar{r}^{b]}{}_{d]} &= -2n^{[a}n_{[c}\bar{r}^{b]}{}_{d]} + 2\gamma^{[a}{}_{[c}\bar{r}^{b]}{}_{d]} && (99)\\ &= -2n^{[a}n_{[c}\left\{\frac{1}{12}\gamma^{b]}{}_{d]}\left(3\rho + S\right) + \breve{S}^{b]}{}_{d]}\right\} + 2\gamma^{[a}{}_{[c}\left\{\frac{1}{4}n^{b]}n_{d]}\left(3\rho + S\right) + n^{b]}P_{d]} + P^{b]}n_{d]} + \frac{1}{12}\gamma^{b]}{}_{d]}\left(3\rho + S\right) + \breve{S}^{b]}{}_{d]}\right\} && (100)\\ &= 4n^{[a}n_{[c}\left\{-\frac{1}{24}\gamma^{b]}{}_{d]}\left(3\rho + S\right) - \frac{1}{2}\breve{S}^{b]}{}_{d]}\right\}\\ &\quad + 4n^{[a}n_{[c}\left\{\frac{1}{8}\gamma^{b]}{}_{d]}\left(3\rho + S\right)\right\} + 2n^{[a|}\epsilon^{f}{}_{cd}\left(-\frac{1}{2}\epsilon^{gh}{}_{f}\gamma^{|b]}{}_{g}P_{h}\right) + 2\epsilon^{ab}{}_{e}n_{[c|}\left(-\frac{1}{2}\epsilon^{e}{}_{gh}\gamma^{g}{}_{|d]}P^{h}\right)\\ &\quad + \epsilon^{ab}{}_{e}\epsilon^{f}{}_{cd}\left\{\frac{1}{12}\gamma^{e}{}_{f}\left(3\rho + S\right)\right\} + \epsilon^{ab}{}_{e}\epsilon^{f}{}_{cd}\left(\frac{1}{2}\epsilon^{e}{}_{gh}\epsilon^{ij}{}_{f}\gamma^{g}{}_{i}\breve{S}^{h}{}_{j}\right) && (101)\\ &= 4n^{[a}n_{[c}\left\{\frac{1}{12}\gamma^{b]}{}_{d]}\left(3\rho + S\right) - \frac{1}{2}\breve{S}^{b]}{}_{d]}\right\} + 2n^{[a}\epsilon^{|f|}{}_{cd}\left(\frac{1}{2}\epsilon^{b]}{}_{fh}P^{h}\right) + 2\epsilon^{ab}{}_{e}n_{[c}\left(-\frac{1}{2}\epsilon^{e}{}_{d]h}P^{h}\right)\\ &\quad + \epsilon^{ab}{}_{e}\epsilon^{f}{}_{cd}\left\{\frac{1}{12}\gamma^{e}{}_{f}\left(3\rho + S\right) - \frac{1}{2}\breve{S}^{e}{}_{f}\right\} && (102)\end{aligned}$$

2.11.4 Weyl Tensor

The Weyl tensor is decomposed into

$$c^{ab}{}_{cd} = 4n^{[a}n_{[c}E^{b]}{}_{d]} + 2n^{[a}\epsilon^{|f|}{}_{cd}B^{b]}{}_{f} + 2\epsilon^{ab}{}_{e}n_{[c}\mathfrak{C}^{e}{}_{d]} + \epsilon^{ab}{}_{e}\epsilon^{f}{}_{cd}\mathfrak{D}^{e}{}_{f}. \quad (103)$$

The traceless condition for the Weyl tensor,

$$
\begin{aligned}
0 &= c^{ab}{}_{cb} && (104)\\
&= 4n^{[a}n_{[c}E^{b]}{}_{b]} + 2n^{[a}\epsilon^{|f|}{}_{cb}B^{b]}{}_{f} + 2\epsilon^{ab}{}_{e}n_{[c}\mathfrak{C}^{e}{}_{b]} + \epsilon^{ab}{}_{e}\epsilon^{f}{}_{cb}\mathfrak{D}^{e}{}_{f} && (105)\\
&= n^{a}n_{c}E - E^{a}{}_{c} + n^{a}\epsilon^{f}{}_{cb}B^{b}{}_{f} + \epsilon^{ab}{}_{e}n_{c}\mathfrak{C}^{e}{}_{b} + \epsilon^{ab}{}_{e}\epsilon^{f}{}_{cb}\mathfrak{D}^{e}{}_{f} && (106)\\
&= n^{a}n_{c}E + n^{a}\epsilon^{f}{}_{cb}\left(\frac{1}{3}\gamma^{b}{}_{f}B + \breve{B}^{b}{}_{f} + \epsilon^{b}{}_{fg}B^{g}\right) + \epsilon^{ab}{}_{e}n_{c}\left(\frac{1}{3}\gamma^{e}{}_{b}\mathfrak{C} + \breve{\mathfrak{C}}^{e}{}_{b} + \epsilon^{e}{}_{bf}\mathfrak{C}^{f}\right) + \epsilon^{ab}{}_{e}\epsilon^{f}{}_{cb}\left(\frac{1}{3}\gamma^{e}{}_{f}\mathfrak{D} + \breve{\mathfrak{D}}^{e}{}_{f} + \epsilon^{e}{}_{fg}\mathfrak{D}^{g}\right)\\
&\quad - \frac{1}{3}\gamma^{a}{}_{c}E - \breve{E}^{a}{}_{c} - \epsilon^{a}{}_{cd}E^{d} && (107)\\
&= n^{a}n_{c}E + 2n^{a}B_{c} - 2\mathfrak{C}^{a}n_{c} + \frac{2}{3}\gamma^{a}{}_{c}\mathfrak{D} - \breve{\mathfrak{D}}^{a}{}_{c} + \epsilon^{a}{}_{cd}\mathfrak{D}^{d} - \frac{1}{3}\gamma^{a}{}_{c}E - \breve{E}^{a}{}_{c} - \epsilon^{a}{}_{cd}E^{d} && (108)\\
&= n^{a}n_{c}E + 2n^{a}B_{c} - 2\mathfrak{C}^{a}n_{c} + \frac{1}{3}\gamma^{a}{}_{c}\left(2\mathfrak{D} - E\right) - \breve{E}^{a}{}_{c} - \breve{\mathfrak{D}}^{a}{}_{c} + \epsilon^{a}{}_{cd}\left(\mathfrak{D}^{d} - E^{d}\right), && (109)
\end{aligned}
$$

implies

$$
\begin{aligned}
E &= 0, && (110)\\
B_{a} &= 0, && (111)\\
\mathfrak{C}_{a} &= 0, && (112)\\
\mathfrak{D} &= 0, && (113)\\
\breve{\mathfrak{D}}_{ab} &= -\breve{E}_{ab}, && (114)\\
\mathfrak{D}^{a} &= E^{a}. && (115)
\end{aligned}
$$

The first Bianchi identity,

$$
0 = r^{a}{}_{[bcd]} \Longleftrightarrow 0 = r^{ab}{}_{cd}\varepsilon_{b}{}^{cdg} \qquad (116)
$$

$$
\begin{aligned}
0 &= r^{ab}{}_{cd}\varepsilon_{b}{}^{cdg} && (117)\\
&= \left(\frac{1}{6}r\delta^{[a}{}_{[c}\delta^{b]}{}_{d]} + 2\delta^{[a}{}_{[c}\bar{r}^{b]}{}_{d]} + c^{ab}{}_{cd}\right)\varepsilon_{b}{}^{cdg} && (118)\\
&= c^{ab}{}_{cd}\varepsilon_{b}{}^{cdg} && (119)\\
&= \left(4n^{[a}n_{c}E^{b]}{}_{d} + 2n^{[a}\epsilon^{|f|}{}_{cd}B^{b]}{}_{f} + 2\epsilon^{ab}{}_{e}n_{c}\mathfrak{C}^{e}{}_{d} + \epsilon^{ab}{}_{e}\epsilon^{f}{}_{cd}\mathfrak{D}^{e}{}_{f}\right)\varepsilon_{b}{}^{cdg} && (120)\\
&= -4n^{[a}E^{b]}{}_{d}\epsilon_{b}{}^{dg} + 4n^{[a}B^{b]}{}_{f}\left(-n_{b}\gamma^{fg} + n^{g}\gamma^{f}{}_{b}\right) - 2\epsilon^{ab}{}_{e}\mathfrak{C}^{e}{}_{d}\epsilon_{b}{}^{dg} + 2\epsilon^{ab}{}_{e}\left(-n_{b}\gamma^{fg} + n^{g}\gamma^{f}{}_{b}\right)\mathfrak{D}^{e}{}_{f} && (121)\\
&= -2n^{a}E^{b}{}_{d}\epsilon_{b}{}^{dg} - 2B^{ag} + 2n^{a}n^{g}B - 2\gamma^{ag}\mathfrak{C} + 2\mathfrak{C}^{ga} + 2\epsilon^{ab}{}_{e}n^{g}\mathfrak{D}^{e}{}_{b} && (122)\\
&= 2n^{a}n^{g}B - 2n^{a}\epsilon^{b}{}_{df}E^{f}\epsilon_{b}{}^{dg} + 2\epsilon^{ab}{}_{e}n^{g}\epsilon^{e}{}_{bf}E^{f} - \frac{2}{3}\gamma^{ab}B - 2\breve{B}^{ag} - 2\gamma^{ag}\mathfrak{C} + \frac{2}{3}\gamma^{ag}\mathfrak{C} + 2\breve{\mathfrak{C}}^{ag} && (123)\\
&= 2n^{a}n^{g}B - 4n^{a}E^{g} - 4E^{a}n^{g} - \frac{2}{3}\gamma^{ab}\left(B + 2\mathfrak{C}\right) + 2\left(-\breve{B}^{ag} + \breve{\mathfrak{C}}^{ag}\right) && (124)
\end{aligned}
$$

implies

$$
\begin{aligned}
B &= 0, && (125)\\
E^{a} &= 0, && (126)\\
\mathfrak{C} &= 0, && (127)\\
\breve{\mathfrak{C}}_{ab} &= \breve{B}_{ab}. && (128)
\end{aligned}
$$

As a result,

$$
c^{ab}{}_{cd} = 4n^{[a}n_{[c}\breve{E}^{b]}{}_{d]} + 2n^{[a}\epsilon^{|f|}{}_{cd}\breve{B}^{b]}{}_{f} + 2\epsilon^{ab}{}_{e}n_{[c}\breve{B}^{e}{}_{d]} - \epsilon^{ab}{}_{e}\epsilon^{f}{}_{cd}\breve{E}^{e}{}_{f}, \qquad (129)
$$

where $\breve{E}$ is the electric part and $\breve{B}$ is the magnetic part of the Weyl tensor.

2.11.5 Riemann Tensor

$$
\begin{aligned}
r^{ab}{}_{cd} = {} & 4n^{[a}n_{[c}\left\{\frac{1}{6}\gamma^{b]}{}_{d]}\left(\rho + S\right) - \frac{1}{2}\breve{S}^{b]}{}_{d]} + \breve{E}^{b]}{}_{d]}\right\} + 2n^{[a}\epsilon^{|f|}{}_{cd}\left(\breve{B}^{b]}{}_{f} + \frac{1}{2}\epsilon^{b]}{}_{fg}P^{g}\right)\\
& + 2\epsilon^{ab}{}_{e}n_{[c}\left(\breve{B}^{e}{}_{d]} - \frac{1}{2}\epsilon^{e}{}_{d]g}P^{g}\right) + \epsilon^{ab}{}_{e}\epsilon^{f}{}_{cd}\left\{\frac{1}{3}\gamma^{e}{}_{f}\rho - \frac{1}{2}\breve{S}^{e}{}_{f} - \breve{E}^{e}{}_{f}\right\} \qquad (130)
\end{aligned}
$$

The number of components for quantities are summarized in

- ρ: 1
- P_a: 3
- S: 1
- $\breve{S}_{ab}$: 5
- $\breve{E}_{ab}$: 5
- $\breve{B}_{ab}$: 5
- $c^a{}_{bcd}$: 10
- $r^a{}_{bcd}$: 20

3 Covariant Derivatives

3.1 Acceleration and Shape Operator

To decompose $\nabla_b n^a$, we observe that

$$\begin{aligned} n_a \nabla_b n^a &= \nabla_b \left(n^a n_a\right) - n^a \nabla_b n_a && (131)\\ &= \nabla_b \left(-1\right) - n_a \nabla_b n^a && (132)\\ &= \frac{1}{2} \nabla_b \left(-1\right) && (133)\\ &= 0. && (134) \end{aligned}$$

Therefore, the covariant derivative of the normal vector is decomposition into

$$\nabla_b n^a = -A^a n_b + K^a{}_b, \tag{135}$$

where A is the 4-acceleration and K is the shape operator (also known as the Weingarten map, the extrinsic curvature, or the second fundamental form) given by

$$\begin{aligned} A^a &= n^b \nabla_b n^a, && (136)\\ K^a{}_b &= \gamma^c{}_b \nabla_c n^a. && (137) \end{aligned}$$

We consider the torsion free condition of the Levi-Civita connection for t as

$$\begin{aligned} 0 &= \nabla_{[a} \nabla_{b]} t && (138)\\ &= -\nabla_{[a} \left(N^{-1} n_{b]}\right) && (139)\\ &= -N^{-1} \nabla_{[a} n_{b]} - N^{-2} n_{[a} \nabla_{b]} N && (140)\\ &= -N^{-1} \left(\nabla_{[a} n_{b]} + n_{[a} \nabla_{b]} \ln N\right). && (141)\\ &= -N^{-1} \left\{\nabla_{[a} n_{b]} + n_{[a} \left(-n^c n_{b]} + \gamma^c{}_{b]}\right) \nabla_c \ln N\right\}. && (142)\\ &= -N^{-1} \left\{\nabla_{[a} n_{b]} + n_{[a} \gamma^c{}_{b]} \nabla_c \ln N\right\}. && (143) \end{aligned}$$

It implies

$$\begin{aligned} \nabla_{[b} n_{a]} &= n_{[a} \gamma^c{}_{b]} \nabla_c \ln N && (144)\\ &= n_{[a} A_{b]} + K_{[ab]}. && (145) \end{aligned}$$

We find that

$$\begin{aligned} A_a &= \gamma^b{}_a \nabla_b \ln N, && (146)\\ K_{[ab]} &= 0. && (147) \end{aligned}$$

3.2 Temporal and Spatial Derivative

For spatial tensor X, we define temporal derivative $\mathbb{T}$ and spatial derivative $\mathbb{D}$ as

$$\begin{aligned} \mathbb{T} X &= \perp \left(n^a \nabla_a X\right), && (148)\\ \mathbb{D}_a X &= \perp \left(\nabla_a X\right), && (149) \end{aligned}$$

where $\perp$ is the projection operator into Σ_t for all indices of the operand. For the induced metric and the induced Levi-Civita tensor,

$$\mathbb{T}\gamma_{ab} = \perp\left(n^c \nabla_c \gamma_{ab}\right) \tag{150}$$
$$= \perp\left(n^c \nabla_c \left(n_a n_b\right)\right) \tag{151}$$
$$= \perp\left(n_a n^c \nabla_c n_b + n_b n^c \nabla_c n_a\right) \tag{152}$$
$$= 0, \tag{153}$$
$$\mathbb{D}_c \gamma_{ab} = \perp\left(\nabla_c \gamma_{ab}\right) \tag{154}$$
$$= \perp\left(n_a \nabla_c n_b + n_b \nabla_c n_a\right) \tag{155}$$
$$= 0, \tag{156}$$
$$\mathbb{T}\epsilon_{abc} = \perp\left(n^d \nabla_d \epsilon_{abc}\right) \tag{157}$$
$$= \perp\left(n^d \nabla_d \left(n^e \varepsilon_{eabc}\right)\right) \tag{158}$$
$$= \perp\left(\varepsilon_{eabc} n^d \nabla_d n^e\right) \tag{159}$$
$$= \perp\left(\varepsilon_{eabc} A^e\right) \tag{160}$$
$$= 0, \tag{161}$$
$$\mathbb{D}_d \epsilon_{abc} = \perp\left(\nabla_d \left(n^e \varepsilon_{eabc}\right)\right) \tag{162}$$
$$= \perp\left(\varepsilon_{eabc} \nabla_d n^e\right) \tag{163}$$
$$= \perp\left(\varepsilon_{eabc}\left(-A^e n_d + K^e{}_d\right)\right) \tag{164}$$
$$= 0. \tag{165}$$

Note that any metric duals to γ and ϵ are also vanished by $\mathbb{T}$ and $\mathbb{D}$.

$\mathbb{D}$ is torsion free because, for a scalar X,

$$\mathbb{D}_b \mathbb{D}_a X = \perp\left\{\nabla_b \left(\nabla_a X - n_a n^c \nabla_c X\right)\right\} \tag{166}$$
$$= \perp\left\{\nabla_b \nabla_a X - \left(\nabla_b n_a\right) n^c \nabla_c X\right\} \tag{167}$$
$$= \perp\left(\nabla_b \nabla_a X - K_{ab} n^c \nabla_c X\right), \tag{168}$$
$$\mathbb{D}_{[a} \mathbb{D}_{b]} X = 0. \tag{169}$$

3.3 Orthogonal Decomposition of Covariant Derivatives

For a scalar X,

$$\nabla_a X = -n_a n^b \nabla_b X + \gamma^b{}_a \nabla_b X \tag{170}$$
$$= -n_a \mathbb{T}X + \mathbb{D}X. \tag{171}$$

For a spatial linear form Y,

$$\nabla_b Y_a = -n_a n^c \nabla_b Y_c - n_b \mathbb{T} Y_a + \mathbb{D}_b Y_a \tag{172}$$
$$= -n_a \left\{\nabla_b \left(n^c Y_c\right) - Y_c \nabla_b n^c\right\} - n_b \mathbb{T} Y_a + \mathbb{D}_b Y_a \tag{173}$$
$$= n_a \left(-A^c n_b + K^c{}_b\right) Y_c - n_b \mathbb{T} Y_a + \mathbb{D}_b Y_a \tag{174}$$
$$= -n_a n_b A^c Y_c + n_a K^c{}_b Y_c - n_b \mathbb{T} Y_a + \mathbb{D}_b Y_a. \tag{175}$$

For a spatial rank (0,2) tensor Z,

$$\nabla_c Z_{ab} = n_a n_b n^d n^e \nabla_c Z_{de} - n_a n^d \gamma^e{}_b \nabla_c Z_{de} - \gamma^d{}_a n_b n^e \nabla_c Z_{de} - n_c \mathbb{T} Z_{ab} + \mathbb{D}_c Z_{ab} \tag{176}$$
$$= -n_a n_b n^e Z_{de} \nabla_c n^d + n_a \gamma^e{}_b Z_{de} \nabla_c n^d + \gamma^d{}_a n_b Z_{de} \nabla_c n^e - n_c \mathbb{T} Z_{ab} + \mathbb{D}_c Z_{ab} \tag{177}$$
$$= -n_a Z_{db} A^d n_c + n_a Z_{db} K^d{}_c - Z_{ad} A^d n_b n_c + Z_{ad} n_b K^d{}_c - n_c \mathbb{T} Z_{ab} + \mathbb{D}_c Z_{ab}. \tag{178}$$

3.4 Orthogonal Decomposition of the Ricci Identity

The Ricci identity for the normal vector is given by

$$r^a{}_{dbc} n^d = 2\nabla_{[b} \nabla_{c]} n^a. \tag{179}$$

For the decomposition of the right-hand side, we start from

$$
\begin{aligned}
\nabla_c \nabla_b n^a &= \nabla_c \left(-A^a n_b + K^a{}_b\right) && (180)\\
&= -A^a \nabla_c n_b - n_b \nabla_c A^a + \nabla_c K^a{}_b && (181)\\
&= -A^a \left(-A_b n_c + K_{bc}\right) - n_b \left(-n^a n_c A^d A_d + n^a K^d{}_c A_d - n_c \mathbb{T} A^a + \mathbb{D}_c A^a\right)\\
&\quad - n^a K_{db} A^d n_c + n^a K_{db} K^d{}_c - K^a{}_d A^d n_b n_c + K^a{}_d n_b K^d{}_c - n_c \mathbb{T} K^a{}_b + \mathbb{D}_c K^a{}_b && (182)\\
&= n^a n_b n_c A^d A_d - n^a n_b K^d{}_c A_d - n^a n_c K_{db} A^d + n^a K_{db} K^d{}_c + n_b n_c \mathbb{T} A^a - n_b n_c K^a{}_d A^d - n_b \mathbb{D}_c A^a + n_b K^a{}_d K^d{}_c\\
&\quad + n_c A^a A_b - n_c \mathbb{T} K^a{}_b - A^a K_{bc} + \mathbb{D}_c K^a{}_b. && (183)
\end{aligned}
$$

Then,

$$
\begin{aligned}
2\nabla_{[b}\nabla_{c]} n^a &= 2n_{[b}\left(-\mathbb{T}K^a{}_{c]} + \mathbb{D}_{c]}A^a - K^a{}_{|d|}K^d{}_{c]} + A^a A_{c]}\right) + 2\mathbb{D}_{[b}K^a{}_{c]} && (184)\\
&= 2n_{[b}\left(-\mathbb{T}K^a{}_{c]} + \mathbb{D}_{c]}A^a - K^a{}_{|d|}K^d{}_{c]} + A^a A_{c]}\right) + \epsilon^d{}_{bc}\left(\epsilon^{ef}{}_d \mathbb{D}_e K^a{}_f\right) && (185)\\
&= 2n_{[b}\left(\mathfrak{A}^a{}_{c]}\right) + \epsilon^d{}_{bc}\left(\mathfrak{B}^a{}_d\right). && (186)
\end{aligned}
$$

By the irreducible decomposition, we get

$$
\begin{aligned}
\mathfrak{A} &= -\mathbb{T}K + \mathbb{D}_a A^a - \frac{1}{3}K^2 - \breve{K}_{ab}\breve{K}^{ab} + A_a A^a, && (187)\\
\breve{\mathfrak{A}}_{ab} &= -\mathbb{T}\breve{K}_{ab} + \mathbb{D}_{<a}A_{b>} - \frac{2}{3}K\breve{K}_{ab} - \breve{K}_{c<a}\breve{K}_{b>}{}^c + A_{<a}A_{b>}, && (188)\\
\mathfrak{A}^a &= -\frac{1}{2}\epsilon^{abc}\mathbb{D}_b A_c, && (189)\\
\mathfrak{B} &= 0, && (190)\\
\breve{\mathfrak{B}}_{ab} &= \epsilon_{cd<a}\mathbb{D}^c \breve{K}_{b>}{}^d, && (191)\\
\mathfrak{B}^a &= \frac{1}{3}\mathbb{D}^a K - \frac{1}{2}\mathbb{D}_b \breve{K}^{ab}. && (192)
\end{aligned}
$$

where $\langle\cdot\rangle$ on indices is defined by

$$
X_{<ab>} = \left(\gamma^{(c}{}_a \gamma^{d)}{}_b - \frac{1}{3}\gamma_{ab}\gamma^{cd}\right) X_{cd}. \qquad (193)
$$

From eq. (130), we obtain

$$
r^{ab}{}_{cd} n_b = 2n_{[c}\left\{\frac{1}{6}\gamma^a{}_{d]}\left(\rho + S\right) - \frac{1}{2}\breve{S}^a{}_{d]} + \breve{E}^a{}_{d]}\right\} + \epsilon^{|f|}{}_{cd}\left(\breve{B}^a{}_f + \frac{1}{2}\epsilon^a{}_{fh}P^h\right). \qquad (194)
$$

Therefore,

$$
\begin{aligned}
\frac{1}{2}\left(\rho + S\right) &= -\mathbb{T}K + \mathbb{D}_a A^a - \frac{1}{3}K^2 - \breve{K}_{ab}\breve{K}^{ab} + A_a A^a, && (195)\\
-\frac{1}{2}\breve{S}_{ab} + \breve{E}_{ab} &= -\mathbb{T}\breve{K}_{ab} + \mathbb{D}_{<a}A_{b>} - \frac{2}{3}K\breve{K}_{ab} - \breve{K}_{c<a}\breve{K}_{b>}{}^c + A_{<a}A_{b>}, && (196)\\
0 &= -\frac{1}{2}\epsilon^{abc}\mathbb{D}_b A_c, && (197)\\
0 &= 0, && (198)\\
\breve{B}_{ab} &= \epsilon_{cd<a}\mathbb{D}^c \breve{K}_{b>}{}^d, && (199)\\
\frac{1}{2}P^a &= \frac{1}{3}\mathbb{D}^a K - \frac{1}{2}\mathbb{D}_b \breve{K}^{ab}, && (200)
\end{aligned}
$$

where the right-hand side of eq. (197) vanishes because $A_a = \mathbb{D}_a \ln N$ and $\mathbb{D}$ is torsion free. Note that eqs. (195) and (196) are evolution equations for K and $\breve{K}$, respectively, and eqs. (199) and (200) are constraint equations. System of these equations is incomplete because there is no an evolution equation for $\breve{E}$. We will find it in the second Bianchi identity.

3.5 Orthogonal Decomposition of the Second Bianchi Identity

The second Bianchi identity can be expressed by the spacetime Levi-Civita tensor given by

$$
0 = \nabla_{[e} r^a{}_{|b|cd]} \Longleftrightarrow 0 = \nabla^d\left(\frac{1}{2}\epsilon^{gh}{}_{cd} r^{ab}{}_{gh}\right). \qquad (201)
$$

For simplicity, we introduce auxiliary variables as

$$r^{ab}{}_{cd} = 4n^{[a}n_{[c}\mathfrak{A}^{b]}{}_{d]} + 2n^{[a}\epsilon^{|f|}{}_{cd}\mathfrak{B}^{b]}{}_{f} + 2\epsilon^{ab}{}_{e}n_{[c}\mathfrak{C}^{e}{}_{d]} + \epsilon^{ab}{}_{e}\epsilon^{f}{}_{cd}\mathfrak{D}^{e}{}_{f}, \tag{202}$$

where

$$\mathfrak{A}_{ab} = \frac{1}{6}\gamma_{ab}\left(\rho + S\right) - \frac{1}{2}\breve{S}_{ab} + \breve{E}_{ab}, \tag{203}$$

$$\mathfrak{B}_{ab} = \breve{B}_{ab} + \frac{1}{2}\epsilon^{c}{}_{ab}P_{c}, \tag{204}$$

$$\mathfrak{C}_{ab} = \breve{B}_{ab} - \frac{1}{2}\epsilon^{c}{}_{ab}P_{c}, \tag{205}$$

$$\mathfrak{D}_{ab} = \frac{1}{3}\gamma_{ab}\rho - \frac{1}{2}\breve{S}_{ab} - \breve{E}_{ab}. \tag{206}$$

Then, its Hodge dual becomes

$$\frac{1}{2}\epsilon^{gh}{}_{cd}r^{ab}{}_{gh} = \frac{1}{2}\epsilon^{gh}{}_{cd}\left(4n^{[a}n_{g}\mathfrak{A}^{b]}{}_{h} + 2n^{[a}\epsilon^{|f|}{}_{gh}\mathfrak{B}^{b]}{}_{f} + 2\epsilon^{ab}{}_{e}n_{g}\mathfrak{C}^{e}{}_{h} + \epsilon^{ab}{}_{e}\epsilon^{f}{}_{gh}\mathfrak{D}^{e}{}_{f}\right) \tag{207}$$

$$= 2\epsilon^{h}{}_{cd}n^{[a}\mathfrak{A}^{b]}{}_{h} + \left(-n_{c}\gamma^{f}{}_{d} + n_{d}\gamma^{f}{}_{c}\right)2n^{[a}\mathfrak{B}^{b]}{}_{f} + \epsilon^{h}{}_{cd}\epsilon^{ab}{}_{e}\mathfrak{C}^{e}{}_{h} + \left(-n_{c}\gamma^{f}{}_{d} + n_{d}\gamma^{f}{}_{c}\right)\epsilon^{ab}{}_{e}\mathfrak{D}^{e}{}_{f} \tag{208}$$

$$= -4n^{[a}n_{[c}\mathfrak{B}^{b]}{}_{d]} + 2n^{[a}\epsilon^{|f|}{}_{cd}\mathfrak{A}^{b]}{}_{f} - 2\epsilon^{ab}{}_{e}n_{[c}\mathfrak{D}^{e}{}_{d]} + \epsilon^{ab}{}_{e}\epsilon^{f}{}_{cd}\mathfrak{C}^{e}{}_{f}. \tag{209}$$

The second Bianchi identity is decomposed into

$$0 = \nabla^{d}\left(\frac{1}{2}\epsilon^{gh}{}_{cd}r^{ab}{}_{gh}\right) \tag{210}$$

$$= \nabla^{d}\left(-4n^{[a}n_{[c}\mathfrak{B}^{b]}{}_{d]} + 2n^{[a}\epsilon^{|f|}{}_{cd}\mathfrak{A}^{b]}{}_{f} - 2\epsilon^{ab}{}_{e}n_{[c}\mathfrak{D}^{e}{}_{d]} + \epsilon^{ab}{}_{e}\epsilon^{f}{}_{cd}\mathfrak{C}^{e}{}_{f}\right) \tag{211}$$

$$\begin{aligned}
&= -4\left(-n^{d}A^{[a} + K^{d[a}\right)n_{[c}\mathfrak{B}^{b]}{}_{d]} - 4n^{[a}\left(-n^{|d|}A_{[c} + K^{|d|}{}_{[c}\right)\mathfrak{B}^{b]}{}_{d]} - 4n^{[a}n_{[c}\nabla^{|d|}\mathfrak{B}^{b]}{}_{d]} \\
&\quad + 2\left(-n^{d}A^{[a} + K^{d[a}\right)\epsilon^{|f|}{}_{cd}\mathfrak{A}^{b]}{}_{f} + 2n^{[a|}\left(-n^{d}A^{e} + K^{de}\right)\epsilon_{e}{}^{f}{}_{cd}\mathfrak{A}^{|b]}{}_{f} + 2n^{[a|}\epsilon^{f}{}_{cd}\nabla^{d}\mathfrak{A}^{|b]}{}_{f} \\
&\quad - 2\left(-n^{d}A^{f} + K^{df}\right)\epsilon_{f}{}^{ab}{}_{e}n_{[c}\mathfrak{D}^{e}{}_{d]} - 2\epsilon^{ab}{}_{e}\left(-n^{d}A_{[c} + K^{d}{}_{[c}\right)\mathfrak{D}^{e}{}_{d]} - 2\epsilon^{ab}{}_{e}n_{[c}\nabla^{d}\mathfrak{D}^{e}{}_{d]} \\
&\quad + \left(-n^{d}A^{g} + K^{dg}\right)\epsilon_{g}{}^{ab}{}_{e}\epsilon^{f}{}_{cd}\mathfrak{C}^{e}{}_{f} + \epsilon^{ab}{}_{e}\left(-n^{d}A^{g} + K^{dg}\right)\epsilon_{g}{}^{f}{}_{cd}\mathfrak{C}^{e}{}_{f} + \epsilon^{ab}{}_{e}\epsilon^{f}{}_{cd}\nabla^{d}\mathfrak{C}^{e}{}_{f}
\end{aligned} \tag{212}$$

$$\begin{aligned}
&= 2A^{[a}\mathfrak{B}^{b]}{}_{c} - 2K^{d[a}n_{c}\mathfrak{B}^{b]}{}_{d} - 2n^{[a}K^{|d|}{}_{c}\mathfrak{B}^{b]}{}_{d} + 2n^{[a}K\mathfrak{B}^{b]}{}_{c} - 2n^{[a}n_{c}\nabla^{|d|}\mathfrak{B}^{b]}{}_{d} + 2n^{[a}n_{d}\nabla^{|d|}\mathfrak{B}^{b]}{}_{c} \\
&\quad + 2K^{d[a}\epsilon^{|f|}{}_{cd}\mathfrak{A}^{b]}{}_{f} + 2n^{[a|}A^{e}\epsilon_{e}{}^{f}{}_{c}\mathfrak{A}^{|b]}{}_{f} + 2n^{[a|}\epsilon^{f}{}_{cd}\nabla^{d}\mathfrak{A}^{|b]}{}_{f} \\
&\quad - \left(-A^{f}\mathfrak{D}^{e}{}_{c} + K^{df}n_{c}\mathfrak{D}^{e}{}_{d}\right)2n^{[a}\epsilon^{b]}{}_{ef} - \epsilon^{ab}{}_{e}K^{d}{}_{c}\mathfrak{D}^{e}{}_{d} + \epsilon^{ab}{}_{e}K\mathfrak{D}^{e}{}_{c} - \epsilon^{ab}{}_{e}n_{c}\nabla^{d}\mathfrak{D}^{e}{}_{d} + \epsilon^{ab}{}_{e}n_{d}\nabla^{d}\mathfrak{D}^{e}{}_{c} \\
&\quad + K^{dg}2n^{[a}\epsilon_{g}{}^{b]}{}_{e}\epsilon^{f}{}_{cd}\mathfrak{C}^{e}{}_{f} + \epsilon^{ab}{}_{e}A^{g}\epsilon_{g}{}^{f}{}_{c}\mathfrak{C}^{e}{}_{f} + \epsilon^{ab}{}_{e}\epsilon^{f}{}_{cd}\nabla^{d}\mathfrak{C}^{e}{}_{f}
\end{aligned} \tag{213}$$

$$\begin{aligned}
&= 2n^{[a|}\Big\{n_{c}\left(-\nabla^{d}\mathfrak{B}^{|b]}{}_{d} + \epsilon^{|b]}{}_{de}K^{df}\mathfrak{D}^{e}{}_{f}\right) \\
&\qquad -K^{d}{}_{c}\mathfrak{B}^{|b]}{}_{d} + K\mathfrak{B}^{|b]}{}_{c} + n_{d}\nabla^{d}\mathfrak{B}^{|b]}{}_{c} + A^{e}\epsilon_{e}{}^{f}{}_{c}\mathfrak{A}^{|b]}{}_{f} + \epsilon^{f}{}_{cd}\nabla^{d}\mathfrak{A}^{|b]}{}_{f} + \epsilon^{|b]}{}_{ef}A^{f}\mathfrak{D}^{e}{}_{c} + K^{dg}\epsilon_{g}{}^{|b]}{}_{e}\epsilon^{f}{}_{cd}\mathfrak{C}^{e}{}_{f}\Big\} \\
&\quad + \epsilon^{ab}{}_{d}\Big\{n_{c}\left(-\epsilon^{d}{}_{ef}K^{ge}\mathfrak{B}^{f}{}_{g} - \nabla^{e}\mathfrak{D}^{d}{}_{e}\right) \\
&\qquad +\epsilon^{d}{}_{ef}\left(A^{e}\mathfrak{B}^{f}{}_{c} + K^{ge}\epsilon^{h}{}_{cg}\mathfrak{A}^{f}{}_{h}\right) - K^{e}{}_{c}\mathfrak{D}^{d}{}_{e} + K\mathfrak{D}^{d}{}_{c} + n_{e}\nabla^{e}\mathfrak{D}^{d}{}_{c} + A^{g}\epsilon_{g}{}^{f}{}_{c}\mathfrak{C}^{d}{}_{f} + \epsilon^{f}{}_{ce}\nabla^{e}\mathfrak{C}^{d}{}_{f}\Big\}.
\end{aligned} \tag{214}$$

Using eq. (178), we get

$$n^{[a|}\nabla^{d}\mathfrak{B}^{|b]}{}_{d} = n^{[a|}\left(\mathfrak{B}^{|b]}{}_{d}A^{d} + \mathbb{D}^{d}\mathfrak{B}^{|b]}{}_{d}\right), \tag{215}$$

$$n^{[a|}n_{d}\nabla^{d}\mathfrak{B}^{|b]}{}_{c} = n^{[a|}\left(\mathfrak{B}^{|b]}{}_{d}A^{d}n_{c} + \mathbb{T}\mathfrak{B}^{|b]}{}_{c}\right), \tag{216}$$

$$n^{[a|}\epsilon^{f}{}_{cd}\nabla^{d}\mathfrak{A}^{|b]}{}_{f} = n^{[a|}\epsilon^{f}{}_{cd}\mathbb{D}^{d}\mathfrak{A}^{|b]}{}_{f}, \tag{217}$$

$$\epsilon^{ab}{}_{d}\nabla^{e}\mathfrak{D}^{d}{}_{e} = \epsilon^{ab}{}_{d}\left(\mathfrak{D}^{d}{}_{e}A^{e} + \mathbb{D}^{e}\mathfrak{D}^{d}{}_{e}\right), \tag{218}$$

$$\epsilon^{ab}{}_{d}n_{e}\nabla^{e}\mathfrak{D}^{d}{}_{c} = \epsilon^{ab}{}_{d}\left(\mathfrak{D}^{d}{}_{e}A^{e}n_{c} + \mathbb{T}\mathfrak{D}^{d}{}_{c}\right), \tag{219}$$

$$\epsilon^{ab}{}_{d}\epsilon^{f}{}_{ce}\nabla^{e}\mathfrak{C}^{d}{}_{f} = \epsilon^{ab}{}_{d}\left(\epsilon^{f}{}_{ce}\mathbb{D}^{e}\mathfrak{C}^{d}{}_{f}\right). \tag{220}$$

Then,

$$
\begin{aligned}
0 &= \nabla^d \left(\frac{1}{2} \epsilon^{gh}{}_{cd} r^{ab}{}_{gh} \right) && (221) \\
&= 2n^{[a|} \Big\{ n_c \left(-\mathbb{D}^d \mathfrak{B}^{|b]}{}_d + \epsilon^{|b]}{}_{de} K^{df} \mathfrak{D}^e{}_f \right) \\
&\qquad - K^d{}_c \mathfrak{B}^{|b]}{}_d + K \mathfrak{B}^{|b]}{}_c + \mathbb{T} \mathfrak{B}^{|b]}{}_c + A^e \epsilon_e{}^f{}_c \mathfrak{A}^{|b]}{}_f + \epsilon^f{}_{cd} \mathbb{D}^d \mathfrak{A}^{|b]}{}_f + \epsilon^{|b]}{}_{ef} A^f \mathfrak{D}^e{}_c + K^{dg} \epsilon_g{}^{|b]}{}_e \epsilon^f{}_{cd} \mathfrak{C}^e{}_f \Big\} \\
&\quad + \epsilon^{ab}{}_d \left\{ n_c \left(-\epsilon^d{}_{ef} K^{ge} \mathfrak{B}^f{}_g - \mathbb{D}^e \mathfrak{D}^d{}_e \right) \right. \\
&\qquad \left. + \epsilon^d{}_{ef} \left(A^e \mathfrak{B}^f{}_c + K^{ge} \epsilon^h{}_{cg} \mathfrak{A}^f{}_h \right) - K^e{}_c \mathfrak{D}^d{}_e + K \mathfrak{D}^d{}_c + \mathbb{T} \mathfrak{D}^d{}_c + A^g \epsilon_g{}^f{}_c \mathfrak{C}^d{}_f + \epsilon^f{}_{ce} \mathbb{D}^e \mathfrak{C}^d{}_f \right\} && (222) \\
&= 2n^{[a} \left(n_c \mathfrak{E}^{b]} + \mathfrak{F}^{b]}{}_c \right) + \epsilon^{ab}{}_d \left(n_c \mathfrak{G}^d + \mathfrak{H}^d{}_c \right), && (223)
\end{aligned}
$$

where $\mathfrak{E}$, $\mathfrak{F}$, $\mathfrak{G}$, and $\mathfrak{H}$ are auxiliary variables. The irreducible decomposition of auxiliary variables are given by

$$
\begin{aligned}
0 &= \mathfrak{E}^a && (224) \\
&= -\mathbb{D}^b \mathfrak{B}^a{}_b + \epsilon^a{}_{bc} K^{bd} \mathfrak{D}^c{}_d && (225) \\
&= -\mathbb{D}^b \left(\breve{B}^a{}_b + \frac{1}{2} \epsilon^{ca}{}_b P_c \right) + \epsilon^a{}_{bc} \left(\frac{1}{3} \gamma^{bd} K + \breve{K}^{bd} \right) \left(\frac{1}{3} \gamma^c{}_d \rho - \left(\frac{1}{2} \breve{S} + \breve{E} \right)^c{}_d \right) && (226) \\
&= -\mathbb{D}_b \breve{B}^{ab} - \frac{1}{2} \epsilon^{abc} \mathbb{D}_b P_c - \epsilon^a{}_{bc} \breve{K}^{bd} \left(\frac{1}{2} \breve{S} + \breve{E} \right)^c{}_d && (227)
\end{aligned}
$$

$$
\begin{aligned}
0 &= \mathfrak{F}_{ab} && (228) \\
&= -K^c{}_b \mathfrak{B}_{ac} + K \mathfrak{B}_{ab} + \mathbb{T} \mathfrak{B}_{ab} + \epsilon^c{}_{bd} A^d \mathfrak{A}_{ac} + \epsilon^c{}_{bd} \mathbb{D}^d \mathfrak{A}_{ac} + \epsilon_{acd} A^d \mathfrak{D}^c{}_b - \epsilon_{acd} \epsilon_{bef} K^{ce} \mathfrak{C}^{df} && (229) \\
&= -\left(\frac{1}{3} \gamma^c{}_b K + K^c{}_b \right) \left(\breve{B}_{ac} + \frac{1}{2} \epsilon^d{}_{ac} P_d \right) + K \left(\breve{B}_{ab} + \frac{1}{2} \epsilon^c{}_{ab} P_c \right) + \mathbb{T} \left(\breve{B}_{ab} + \frac{1}{2} \epsilon^c{}_{ab} P_c \right) \\
&\quad + \epsilon^c{}_{bd} A^d \left(\frac{1}{6} \gamma_{ac} (\rho + S) - \frac{1}{2} \breve{S}_{ac} + \breve{E}_{ac} \right) + \epsilon^c{}_{bd} \mathbb{D}^d \left(\frac{1}{6} \gamma_{ac} (\rho + S) - \frac{1}{2} \breve{S}_{ac} + \breve{E}_{ac} \right) \\
&\quad + \epsilon_{acd} A^d \left(\frac{1}{3} \gamma^c{}_b \rho - \frac{1}{2} \breve{S}^c{}_b - \breve{E}^c{}_b \right) - \epsilon_{acd} \epsilon_{bef} K^{ce} \left(\breve{B}^{df} - \frac{1}{2} \epsilon^{gdf} P_g \right) && (230) \\
&= -\frac{1}{3} K \breve{B}_{ab} - \breve{K}^c{}_b \breve{B}_{ac} - \frac{1}{6} K \epsilon^c{}_{ab} P_c - \frac{1}{2} \epsilon^d{}_{ac} \breve{K}^c{}_b P_d + K \breve{B}_{ab} + \frac{1}{2} \epsilon^c{}_{ab} K P_c + \mathbb{T} \breve{B}_{ab} + \frac{1}{2} \epsilon^c{}_{ab} \mathbb{T} P_c \\
&\quad + \frac{1}{6} \epsilon_{abc} A^c (\rho + S) - \frac{1}{2} \epsilon^c{}_{bd} A^d \breve{S}_{ac} + \epsilon^c{}_{bd} A^d \breve{E}_{ac} + \frac{1}{6} \epsilon^c{}_{ab} \mathbb{D}_c (\rho + S) - \frac{1}{2} \epsilon^c{}_{bd} \mathbb{D}^d \breve{S}_{ac} + \epsilon^c{}_{bd} \mathbb{D}^d \breve{E}_{ac} \\
&\quad + \frac{1}{3} \epsilon_{abc} A^c \rho - \frac{1}{2} \epsilon_{acd} A^d \breve{S}^c{}_b - \epsilon_{acd} A^d \breve{E}^c{}_b + \frac{1}{3} K \breve{B}_{ab} + \frac{1}{6} K \epsilon^c{}_{ab} P_c - \epsilon_{acd} \epsilon_{bef} \breve{K}^{ce} \breve{B}^{df} + \frac{1}{2} \epsilon^c{}_{ab} \breve{K}^d{}_c P_d && (231)
\end{aligned}
$$

$$
\begin{aligned}
0 &= \mathfrak{F} && (232) \\
&= -\breve{K}^{ab} \breve{B}_{ab} + \breve{K}^{ab} \breve{B}^{ab} && (233) \\
&= 0 && (234) \\
0 &= \breve{\mathfrak{F}}_{ab} && (235) \\
&= -\frac{1}{3} K \breve{B}_{ab} - \breve{K}^c{}_{<a} \breve{B}_{b>c} - \frac{1}{2} \epsilon_{cd<a} \breve{K}^c{}_{b>} P^d + K \breve{B}_{ab} + \mathbb{T} \breve{B}_{ab} \\
&\quad - \frac{1}{2} \epsilon_{cd<a} A^c \breve{S}^d{}_{b>} + \epsilon_{cd<a} A^c \breve{E}^d{}_{b>} - \frac{1}{2} \epsilon_{cd<a} \mathbb{D}^c \breve{S}^d{}_{b>} + \epsilon_{cd<a} \mathbb{D}^c \breve{E}^d{}_{b>} \\
&\quad + \frac{1}{2} \epsilon_{cd<a} A^c \breve{S}^d{}_{b>} + \epsilon_{cd<a} A^c \breve{E}^d{}_{b>} + \frac{1}{3} K \breve{B}_{ab} - \epsilon_{acd} \epsilon_{bef} \breve{K}^{ce} \breve{B}^{df} && (236) \\
&= -3 \breve{K}^c{}_{<a} \breve{B}_{b>c} - \frac{1}{2} \epsilon_{cd<a} \breve{K}^c{}_{b>} P^d + K \breve{B}_{ab} + \mathbb{T} \breve{B}_{ab} + 2 \epsilon_{cd<a} A^c \breve{E}^d{}_{b>} + \epsilon_{cd<a} \mathbb{D}^c \left(-\frac{1}{2} \breve{S} + \breve{E} \right)^d{}_{b>} && (237) \\
0 &= \mathfrak{F}^a && (238) \\
&= \frac{1}{2} \epsilon^a{}_{bc} \breve{K}^{bd} \breve{B}^c{}_d - \frac{1}{6} K P^a + \frac{1}{4} \breve{K}^{ab} P_b + \frac{1}{2} K P^a + \frac{1}{2} \mathbb{T} P^a \\
&\quad + \frac{1}{6} A^a (\rho + S) - \frac{1}{2} A^b \left(-\frac{1}{2} \breve{S} + \breve{E} \right)^a{}_b + \frac{1}{6} \mathbb{D}^a (\rho + S) - \frac{1}{2} \mathbb{D}_b \left(-\frac{1}{2} \breve{S} + \breve{E} \right)^{ab} \\
&\quad + \frac{1}{3} A^a \rho + \frac{1}{2} A^b \left(\frac{1}{2} \breve{S} + \breve{E} \right)^a{}_b + \frac{1}{6} K P^a + \frac{1}{2} \breve{K}^{ab} P_b && (239) \\
&= \frac{1}{2} \epsilon^a{}_{bc} \breve{K}^{bd} \breve{B}^c{}_d + \frac{3}{4} \breve{K}^{ab} P_b + \frac{1}{2} K P^a + \frac{1}{2} \mathbb{T} P^a + \frac{1}{6} A^a (3\rho + S) + \frac{1}{6} \mathbb{D}^a (\rho + S) - \frac{1}{2} \mathbb{D}_b \left(-\frac{1}{2} \breve{S} + \breve{E} \right)^{ab} + \frac{1}{2} A^b \breve{S}^a{}_b && (240)
\end{aligned}
$$

$$
\begin{aligned}
0 &= \mathfrak{G}^a & (241)\\
&= -\epsilon^a{}_{bc} K^{db} \mathfrak{B}^c{}_d - \mathbb{D}^b \mathfrak{D}^a{}_b & (242)\\
&= -\epsilon^a{}_{bc} \left(\frac{1}{3}\gamma^{db} K + \breve{K}^{db} \right) \left(\breve{B}^c{}_d + \frac{1}{2}\epsilon^{ec}{}_d P_e \right) - \mathbb{D}_b \left(\frac{1}{3}\gamma^{ab}\rho - \left(\frac{1}{2}\breve{S} + \breve{E} \right)^{ab} \right) & (243)\\
&= \frac{1}{3} K P^a - \epsilon^a{}_{bc} \breve{K}^{bd} \breve{B}^c{}_d - \frac{1}{2}\breve{K}^{ab} P_b - \frac{1}{3}\mathbb{D}^a \rho + \mathbb{D}_b \left(\frac{1}{2}\breve{S} + \breve{E} \right)^{ab} & (244)
\end{aligned}
$$

$$
\begin{aligned}
0 &= \mathfrak{H}_{ab} & (245)\\
&= \epsilon_{acd} \left(A^c \mathfrak{B}^d{}_b + \epsilon^f{}_{be} K^{ec} \mathfrak{A}^d{}_f \right) - K^c{}_b \mathfrak{D}_{ac} + K \mathfrak{D}_{ab} + \mathbb{T}\mathfrak{D}_{ab} + \epsilon_c{}^d{}_b A^c \mathfrak{C}_{ad} + \epsilon^c{}_{bd} \mathbb{D}^d \mathfrak{C}_{ac} & (246)\\
&= \epsilon_{acd} \left(A^c \left(\breve{\mathfrak{B}}^d{}_b + \epsilon^{ed}{}_b \mathfrak{B}_e \right) + \epsilon^f{}_{be} K^{ec} \left(\frac{1}{3}\gamma^d{}_f \mathfrak{A} + \breve{\mathfrak{A}}^d{}_f \right) \right) \\
&\quad - K^c{}_b \left(\frac{1}{3}\gamma_{ac}\mathfrak{D} + \breve{\mathfrak{D}}_{ac} \right) + K \left(\frac{1}{3}\gamma_{ab}\mathfrak{D} + \breve{\mathfrak{D}}_{ab} \right) + \mathbb{T} \left(\frac{1}{3}\gamma_{ab}\mathfrak{D} + \breve{\mathfrak{D}}_{ab} \right) \\
&\quad + \epsilon_c{}^d{}_b A^c \left(\breve{\mathfrak{C}}_{ad} + \epsilon^e{}_{ad} \mathfrak{C}_e \right) + \epsilon^c{}_{bd} \mathbb{D}^d \left(\breve{\mathfrak{C}}_{ac} + \epsilon^e{}_{ac} \mathfrak{C}_e \right) & (247)\\
&= \epsilon_{acd} A^c \breve{\mathfrak{B}}^d{}_b + \gamma_{ab} A^c \mathfrak{B}_c - A_b \mathfrak{B}_a + \frac{2}{9}\gamma_{ab} K \mathfrak{A} - \frac{1}{3} K \breve{\mathfrak{A}}_{ba} - \frac{1}{3}\breve{K}_{ab}\mathfrak{A} + \epsilon_{acd}\epsilon_{bef} \breve{K}^{ce} \breve{\mathfrak{A}}^{df} \\
&\quad - \frac{1}{9}\gamma_{ab} K \mathfrak{D} - \frac{1}{3}\breve{K}_{ab}\mathfrak{D} - \frac{1}{3} K \breve{\mathfrak{D}}_{ab} - \breve{K}^c{}_b \breve{\mathfrak{D}}_{ac} + \frac{1}{3}\gamma_{ab} K \mathfrak{D} + K \breve{\mathfrak{D}}_{ab} + \frac{1}{3}\gamma_{ab}\mathbb{T}\mathfrak{D} + \mathbb{T}\breve{\mathfrak{D}}_{ab} \\
&\quad + \epsilon^d{}_{bc} A^c \breve{\mathfrak{C}}_{ad} + A_a \mathfrak{C}_b - \gamma_{ab} A^c \mathfrak{C}_c + \epsilon^c{}_{bd} \mathbb{D}^d \breve{\mathfrak{C}}_{ac} + \mathbb{D}_a \mathfrak{C}_b - \gamma_{ab} \mathbb{D}^c \mathfrak{C}_c & (248)
\end{aligned}
$$

$$
\begin{aligned}
0 &= \mathfrak{H} & (249)\\
&= 2A^a \mathfrak{B}_a + \frac{2}{3} K \mathfrak{A} - \breve{K}_{ab} \breve{\mathfrak{A}}^{ab} + \frac{2}{3} K \mathfrak{D} - \breve{K}^{ab} \breve{\mathfrak{D}}_{ab} + \mathbb{T}\mathfrak{D} - 2A^a \mathfrak{C}_a - 2\mathbb{D}_a \mathfrak{C}^a & (250)\\
&= A^a P_a + \frac{1}{3} K \left(\rho + S \right) - \breve{K}_{ab} \left(-\frac{1}{2}\breve{S} + \breve{E} \right)^{ab} + \frac{2}{3} K \rho + \breve{K}^{ab} \left(\frac{1}{2}\breve{S} + \breve{E} \right)_{ab} + \mathbb{T}\rho + A^a P_a + \mathbb{D}_a P^a & (251)\\
&= 2A^a P_a + \frac{1}{3} K \left(\rho + S \right) + \breve{K}_{ab} \breve{S}^{ab} + \frac{2}{3} K \rho + \mathbb{T}\rho + \mathbb{D}_a P^a & (252)\\
0 &= \breve{\mathfrak{H}}_{ab} & (253)\\
&= \epsilon_{cd<a} A^c \breve{\mathfrak{B}}^d{}_{b>} - A_{<a} \mathfrak{B}_{b>} - \frac{1}{3} K \breve{\mathfrak{A}}_{ab} - \frac{1}{3}\breve{K}_{ab} \mathfrak{A} + 2\breve{K}_{c<a} \breve{\mathfrak{A}}_{b>}{}^c \\
&\quad - \frac{1}{3}\breve{K}_{ab}\mathfrak{D} - \frac{1}{3}\breve{\mathfrak{D}}_{ab} - \breve{K}^c{}_{<a} \breve{\mathfrak{D}}_{b>c} + K \breve{\mathfrak{D}}_{ab} + \mathbb{T}\breve{\mathfrak{D}}_{ab} \\
&\quad + \epsilon_{cd<a} A^c \breve{\mathfrak{C}}^d{}_{b>} + A_{<a} \mathfrak{C}_{b>} + \epsilon_{cd<a} \mathbb{D}^c \breve{\mathfrak{C}}^d{}_{b>} + \mathbb{D}_{<a} \mathfrak{C}_{b>} & (254)\\
&= 2\epsilon_{cd<a} A^c \breve{B}^d{}_{b>} - A_{<a} P_{b>} - K \left(\frac{1}{6}\breve{S} + \breve{E} \right)_{ab} - \frac{1}{6}\breve{K}_{ab} \left(3\rho + S \right) + \breve{K}_{c<a} \left(-\frac{1}{2}\breve{S} + 3\breve{E} \right)_{b>}{}^{c} \\
&\quad - \mathbb{T} \left(\frac{1}{2}\breve{S} + \breve{E} \right)_{ab} + \epsilon_{cd<a} \mathbb{D}^c \breve{B}^d{}_{b>} - \frac{1}{2}\mathbb{D}_{<a} P_{b>} & (255)\\
0 &= \mathfrak{H}^a & (256)\\
&= \frac{1}{2} A_b \breve{\mathfrak{B}}^{ab} + \frac{1}{2}\epsilon^a{}_{bc} \left(A^b \mathfrak{B}^c + \breve{K}^{bd} \breve{\mathfrak{D}}^c{}_d \right) - \frac{1}{2} A_b \breve{\mathfrak{C}}^{ab} + \frac{1}{2}\epsilon^a{}_{bc} \left(A^b \mathfrak{C}^c \right) - \frac{1}{2}\mathbb{D}_b \breve{\mathfrak{C}}^{ab} + \frac{1}{2}\epsilon^a{}_{bc} \mathbb{D}^b \mathfrak{C}^c & (257)\\
&= -\frac{1}{2}\epsilon^a{}_{bc} \breve{K}^{bd} \left(\frac{1}{2}\breve{S} + \breve{E} \right)^c{}_d - \frac{1}{2}\mathbb{D}_b \breve{B}^{ab} - \frac{1}{4}\epsilon^a{}_{bc} \mathbb{D}^b P^c & (258)
\end{aligned}
$$

3.6 System of Equations

Evolution equations are given by

$$\frac{1}{2}(\rho+S) = -\mathbb{T}K + \mathbb{D}_a A^a - \frac{1}{3}K^2 - \breve{K}_{ab}\breve{K}^{ab} + A_a A^a, \tag{259}$$

$$-\frac{1}{2}\breve{S}_{ab} + \breve{E}_{ab} = -\mathbb{T}\breve{K}_{ab} + \mathbb{D}_{<a}A_{b>} - \frac{2}{3}K\breve{K}_{ab} - \breve{K}_{c<a}\breve{K}_{b>}{}^{c} + A_{<a}A_{b>}, \tag{260}$$

$$\begin{aligned} 0 &= 2\epsilon_{cd<a}A^c\breve{B}^d{}_{b>} - A_{<a}P_{b>} - K\left(\frac{1}{6}\breve{S}+\breve{E}\right)_{ab} - \frac{1}{6}\breve{K}_{ab}(3\rho+S) + \breve{K}_{c<a}\left(-\frac{1}{2}\breve{S}+3\breve{E}\right)_{b>}^{\ \ c} \\ &\quad - \frac{1}{2}\mathbb{T}\breve{S}_{ab} - \mathbb{T}\breve{E}_{ab} + \epsilon_{cd<a}\mathbb{D}^c\breve{B}^d{}_{b>} - \frac{1}{2}\mathbb{D}_{<a}P_{b>}, \end{aligned} \tag{261}$$

$$0 = -3\breve{K}^c{}_{<a}\breve{B}_{b>c} - \frac{1}{2}\epsilon_{cd<a}\breve{K}^c{}_{b>}P^d + K\breve{B}_{ab} + \mathbb{T}\breve{B}_{ab} + 2\epsilon_{cd<a}A^c\breve{E}^d{}_{b>} + \epsilon_{cd<a}\mathbb{D}^c\left(-\frac{1}{2}\breve{S}+\breve{E}\right)^d{}_{b>}, \tag{262}$$

$$0 = 2A^aP_a + \frac{1}{3}K(\rho+S) + \breve{K}_{ab}\breve{S}^{ab} + \frac{2}{3}K\rho + \mathbb{T}\rho + \mathbb{D}^aP_a, \tag{263}$$

$$0 = \epsilon_{abc}\breve{K}^{bd}\breve{B}^c{}_d + \frac{3}{2}\breve{K}_{ab}P^b + KP_a + \mathbb{T}P_a + \frac{1}{3}A_a(3\rho+S) + \frac{1}{3}\mathbb{D}_a(\rho+S) - \mathbb{D}^b\left(-\frac{1}{2}\breve{S}+\breve{E}\right)_{ab} + A^b\breve{S}_{ab}. \tag{264}$$

Constraint equations are given by

$$\frac{1}{2}P^a = \frac{1}{3}\mathbb{D}^aK - \frac{1}{2}\mathbb{D}_b\breve{K}^{ab}, \tag{265}$$

$$\breve{B}_{ab} = \epsilon_{cd<a}\mathbb{D}^c\breve{K}_{b>}{}^d, \tag{266}$$

$$0 = \frac{1}{3}KP_a - \epsilon^c{}_{ab}\breve{K}^{bd}\breve{B}_{cd} - \frac{1}{2}\breve{K}^b{}_aP_b - \frac{1}{3}\mathbb{D}^a\rho + \mathbb{D}^b\left(\frac{1}{2}\breve{S}+\breve{E}\right)_{ab}, \tag{267}$$

$$0 = -\mathbb{D}_b\breve{B}^{ab} - \frac{1}{2}\epsilon^{abc}\mathbb{D}_bP_c - \epsilon^a{}_{bc}\breve{K}^{bd}\left(\frac{1}{2}\breve{S}+\breve{E}\right)^c{}_d. \tag{268}$$

Note that addition of eq. (267) to eq. (264) becomes

$$0 = \breve{K}^b{}_aP^b + \frac{4}{3}KP_a + \mathbb{T}P_a + \frac{1}{3}A_a(3\rho+S) + \frac{1}{3}\mathbb{D}_aS + \mathbb{D}^b\breve{S}_{ab} + A^b\breve{S}_{ab}. \tag{269}$$

4 Application to Numerical Relativity

4.1 Adapted Coordinate

We consider an adapted coordinate $\{x^\mu : \mu = 0,1,2,3\}$ to the foliation Σ_t where $x^0 = t$. The tangent vector $(\partial/\partial t)^a$ along $x^i = \text{const}$ line depends on a choice of spatial coordinate $\{x^i : i = 1,2,3\}$. The orthogonal decomposition of $(\partial/\partial t)^a$ is given by

$$(\partial/\partial t)^a = Nn^a + N^a, \tag{270}$$

where N is the lapse function that determines a time slicing and N^a is the shift vector that indicates a choice of spatial coordinate $\{x^i\}$.

Let us consider Lie derivatives along $(\partial/\partial t)^a$ of covariant spatial tensors. For a scalar X,

$$\begin{aligned} \mathcal{L}_{(\partial/\partial t)}X &= \mathcal{L}_NX + Nn^a\nabla_aX \qquad (271)\\ &= \mathcal{L}_NX + N\mathbb{T}X, \qquad (272)\\ \mathbb{T}X &= N^{-1}\mathcal{L}_{(\partial/\partial t)}X - N^{-1}\mathcal{L}_NX. \qquad (273) \end{aligned}$$

For a linear form Y_a,

$$\begin{aligned} \mathcal{L}_{(\partial/\partial t)}Y_a &= \mathcal{L}_NY_a + Nn^b\nabla_bY_a + Y_b\nabla_a\left(Nn^b\right) \qquad (274)\\ &= \mathcal{L}_NY_a + N\left(n_aA^cY_c + \mathbb{T}Y_a\right) + NY_b\nabla_an^b \qquad (275)\\ &= \mathcal{L}_NY_a + N\left(n_aA^cY_c + \mathbb{T}Y_a - Y_bA^bn_a + Y_bK^b{}_a\right) \qquad (276)\\ &= \mathcal{L}_NY_a + N\left(\mathbb{T}Y_a + Y_bK^b{}_a\right), \qquad (277)\\ \mathbb{T}Y_a &= N^{-1}\mathcal{L}_{(\partial/\partial t)}Y_a - N^{-1}\mathcal{L}_NY_a - Y_bK^b{}_a. \qquad (278) \end{aligned}$$

For a rank (0,2) tensor Z_{ab},

$$\mathcal{L}_{(\partial/\partial t)}Z_{ab} = \mathcal{L}_N Z_{ab} + Nn^c\nabla_c Z_{ab} + Z_{cb}\nabla_a(Nn^c) + Z_{ac}\nabla_b(Nn^c) \tag{279}$$
$$= \mathcal{L}_N Z_{ab} + N\left(n_a Z_{db}A^d + Z_{ad}A^d n_b + \mathbb{T}Z_{ab}\right) + Z_{cb}\nabla_a(Nn^c) + Z_{ac}\nabla_b(Nn^c) \tag{280}$$
$$= \mathcal{L}_N Z_{ab} + N\left(n_a Z_{db}A^d + Z_{ad}A^d n_b + \mathbb{T}Z_{ab} - Z_{cb}A^c n_a + Z_{cb}K^c{}_a - Z_{ac}A^c n_b + Z_{ac}K^c{}_b\right) \tag{281}$$
$$= \mathcal{L}_N Z_{ab} + N\left(\mathbb{T}Z_{ab} + Z_{cb}K^c{}_a + Z_{ac}K^c{}_b\right), \tag{282}$$
$$\mathbb{T}Z_{ab} = N^{-1}\mathcal{L}_{(\partial/\partial t)}Z_{ab} - N^{-1}\mathcal{L}_N Z_{ab} - Z_{cb}K^c{}_a - Z_{ac}K^c{}_b. \tag{283}$$

We rewrite the system of equations as

$$N^{-1}\mathcal{L}_{(\partial/\partial t)}K = N^{-1}\mathcal{L}_N K + \mathbb{D}_a A^a - \frac{1}{3}K^2 - \breve{K}_{ab}\breve{K}^{ab} + A_a A^a - \frac{1}{2}(\rho + S), \tag{284}$$
$$N^{-1}\mathcal{L}_{(\partial/\partial t)}\breve{K}_{ab} = N^{-1}\mathcal{L}_N \breve{K}_{ab} + \mathbb{D}_{<a}A_{b>} + \breve{K}_{c<a}\breve{K}_{b>}{}^c + A_{<a}A_{b>} + \frac{1}{2}\breve{S}_{ab} - \breve{E}_{ab} + \frac{2}{3}\gamma_{ab}\breve{K}^{cd}\breve{K}_{cd}, \tag{285}$$
$$N^{-1}\mathcal{L}_{(\partial/\partial t)}\breve{E}_{ab} = N^{-1}\mathcal{L}_N \breve{E}_{ab} + \epsilon_{cd<a}\mathbb{D}^c\breve{B}^d{}_{b>} - \frac{1}{2}\mathbb{T}\breve{S}_{ab} - \frac{1}{2}\mathbb{D}_{<a}P_{b>} - A_{<a}P_{b>} - \frac{1}{6}\breve{K}_{ab}(3\rho + S) + 2\epsilon_{cd<a}A^c\breve{B}^d{}_{b>} - \frac{1}{3}K\left(\frac{1}{2}\breve{S} + \breve{E}\right)_{ab} + \breve{K}_{c<a}\left(-\frac{1}{2}\breve{S} + 5\breve{E}\right)^c{}_{b>} + \frac{2}{3}\gamma_{ab}\breve{K}^{cd}\breve{E}_{cd}, \tag{286}$$
$$N^{-1}\mathcal{L}_{(\partial/\partial t)}\breve{B}_{ab} = N^{-1}\mathcal{L}_N \breve{B}_{ab} - \epsilon_{cd<a}\mathbb{D}^c\left(-\frac{1}{2}\breve{S} + \breve{E}\right)^d{}_{b>} + \frac{1}{2}\epsilon_{cd<a}\breve{K}^c{}_{b>}P^d + 5\breve{K}^c{}_{<a}\breve{B}_{b>c} - \frac{1}{3}K\breve{B}_{ab} - 2\epsilon_{cd<a}A^c\breve{E}^d{}_{b>} + \frac{2}{3}\gamma_{ab}\breve{K}^{cd}\breve{B}_{cd}, \tag{287}$$
$$N^{-1}\mathcal{L}_{(\partial/\partial t)}\rho = N^{-1}\mathcal{L}_N\rho - \mathbb{D}^a P_a - 2A^a P_a - \frac{1}{3}K(\rho + S) - \breve{K}_{ab}\breve{S}^{ab} - \frac{2}{3}K\rho, \tag{288}$$
$$N^{-1}\mathcal{L}_{(\partial/\partial t)}P_a = N^{-1}\mathcal{L}_N P_a - KP_a - \frac{1}{3}A_a(3\rho + S) - \frac{1}{3}\mathbb{D}_a S - \mathbb{D}^b\breve{S}_{ab} - A^b\breve{S}_{ab} \tag{289}$$
$$\frac{1}{2}P^a = \frac{1}{3}\mathbb{D}^a K - \frac{1}{2}\mathbb{D}_b\breve{K}^{ab}, \tag{290}$$
$$\breve{B}_{ab} = \epsilon_{cd<a}\mathbb{D}^c\breve{K}_{b>}{}^d, \tag{291}$$
$$0 = \frac{1}{3}KP^a - \epsilon^a{}_{bc}\breve{K}^{bd}\breve{B}^c{}_d - \frac{1}{2}\breve{K}^{ab}P_b - \frac{1}{3}\mathbb{D}^a\rho + \mathbb{D}_b\left(\frac{1}{2}\breve{S} + \breve{E}\right)^{ab}, \tag{292}$$
$$0 = -\mathbb{D}_b\breve{B}^{ab} - \frac{1}{2}\epsilon^{abc}\mathbb{D}_b P_c - \epsilon^a{}_{bc}\breve{K}^{bd}\left(\frac{1}{2}\breve{S} + \breve{E}\right)^c{}_d. \tag{293}$$

4.2 Geometrodynamics

We consider an extra 3-dimensional manifold $\hat{\Sigma}$ diffeomorphic to Σ_t where ϕ is their diffeomorphism. The induced metric $\hat{\gamma}_{ab}$ is defined by the pull-back of the spacetime metric as

$$\hat{\gamma}_{ab} \equiv \phi^c{}_a\phi^d{}_b g_{cd} \tag{294}$$
$$= \phi^c{}_a\phi^d{}_b(-n_c n_d + \gamma_{ab}) \tag{295}$$
$$= \phi^c{}_a\phi^d{}_b\left(-N^2(dt)_c(dt)_d + \gamma_{cd}\right) \tag{296}$$
$$= \phi^c{}_a\phi^d{}_b\gamma_{cd}, \tag{297}$$

where $\phi^a{}_b$ is the push-forward for vectors in $\hat{\Sigma}$ or the pull-back operator for linear forms in Σ_t. Note that $\phi^b{}_a(dt)_b = (\phi_*(dt))_a = (d(t\cdot\phi))_a = 0$ because t is constant over Σ_t. We define the inverse metric $\hat{\gamma}^{ab}$ such that

$$\hat{\delta}^a{}_b = \hat{\gamma}^{ac}\hat{\gamma}_{cb}. \tag{298}$$

Observing that

$$\hat{\delta}^a{}_b = \hat{\gamma}^{ac}\hat{\gamma}_{cb} \tag{299}$$
$$= \hat{\gamma}^{ac}\phi^d{}_c\phi^e{}_b\gamma_{de} \tag{300}$$
$$= \left(\hat{\gamma}^{ac}\phi^d{}_c\gamma_{de}\right)\phi^e{}_b \tag{301}$$
$$= \left(\phi^{-1}\right)^a{}_c\phi^c{}_b, \tag{302}$$

we obtain the push-forward or pull-back of the inverse diffeomorphism ϕ^{-1} as

$$\left(\phi^{-1}\right)^a{}_b = \hat{\gamma}^{ac}\phi^d{}_c\gamma_{db}. \tag{303}$$

Then, the metric duals of γ_{ab} is transformed as

$$\gamma^a{}_b \to \left(\phi^{-1}\right)^a{}_c \phi^d{}_b \gamma^c{}_d = \hat{\gamma}^{ae}\phi^f{}_e \gamma_{fc}\phi^d{}_b\gamma^c{}_d \tag{304}$$
$$= \hat{\gamma}^{ae}\phi^f{}_e\gamma_{fd}\phi^d{}_b \tag{305}$$
$$= \hat{\gamma}^{ae}\hat{\gamma}_{eb} \tag{306}$$
$$= \hat{\delta}^a{}_b, \tag{307}$$
$$\gamma^{ab} \to \left(\phi^{-1}\right)^a{}_c \left(\phi^{-1}\right)^b{}_d \gamma^{cd} = \hat{\gamma}^{ae}\phi^f{}_e\gamma_{fc}\hat{\gamma}^{bg}\phi^h{}_g\gamma_{hd}\gamma^{cd}, \tag{308}$$
$$= \hat{\gamma}^{ae}\phi^f{}_e\gamma_{fh}\hat{\gamma}^{bg}\phi^h{}_g \tag{309}$$
$$= \hat{\gamma}^{ae}\hat{\gamma}_{eg}\hat{\gamma}^{bg} \tag{310}$$
$$= \hat{\gamma}^{ab}. \tag{311}$$

We observe that transformations of covariant derivatives of spatial tensors as

$$\mathbb{D}_b X^a \to \left(\phi^{-1}\right)^a{}_c \phi^d{}_b \mathbb{D}_d X^c = \left(\phi^{-1}\right)^a_c \phi^d{}_b \gamma^c{}_e \gamma^f{}_d \nabla_f \left(\phi^e{}_g \hat{X}^g\right) \tag{312}$$
$$= \left(\phi^{-1}\right)^a{}_c \phi^d{}_b \nabla_d \left(\phi^c{}_g \hat{X}^g\right), \tag{313}$$
$$\mathbb{D}_b Y_a \to \phi^c{}_a \phi^d{}_b \mathbb{D}_d Y_c = \phi^c{}_a\phi^d{}_b\gamma^e{}_c\gamma^f{}_d\nabla_f\left(\phi^g{}_e\hat{Y}_g\right) \tag{314}$$
$$= \phi^c{}_a\phi^d{}_b\nabla_d\left(\left(\phi^{-1}\right)^g{}_c\hat{Y}_g\right), \tag{315}$$

where X^a and Y_a are spatial tensors, and $\hat{X}^a$ and $\hat{Y}_a$ are their transformation to $\hat{\Sigma}$, respectively. These suggest that we define a linear connection $\hat{\mathbb{D}}$ in $\hat{\Sigma}$ given by

$$\hat{\mathbb{D}}_b\hat{X}^a = \left(\phi^{-1}\right)^a{}_d\phi^e{}_b\nabla_e\left(\phi^d{}_c\hat{X}^c\right), \tag{316}$$
$$\hat{\mathbb{D}}_b\hat{Y}_a = \phi^d{}_a\phi^e{}_b\nabla_e\left(\left(\phi^{-1}\right)^c{}_d\hat{Y}_c\right), \tag{317}$$

where $\hat{X}^a$ is a vector and $\hat{Y}_a$ is a lienar form in $\hat{\Sigma}$. Surprisingly, it is Levi-Civita connection because

$$\hat{\mathbb{D}}_c\hat{\gamma}_{ab} = \phi^d{}_a\phi^e{}_b\phi^f{}_c\left(\nabla_f\gamma_{de}\right) \tag{318}$$
$$= \phi^d{}_a\phi^e{}_b\phi^f{}_c\left\{\nabla_f\left(n_d n_e\right)\right\} \tag{319}$$
$$= \phi^d{}_a\phi^e{}_b\phi^f{}_c\left\{n_d\nabla_f n_e + n_e\nabla_f n_d\right\} \tag{320}$$
$$= 0, \tag{321}$$
$$\hat{\mathbb{D}}_{[b}\hat{\mathbb{D}}_{a]}\hat{X} = \phi^e{}_{[a}\phi^f{}_{b]}\nabla_f\left\{\left(\phi^{-1}\right)^d{}_e\phi^c{}_d\nabla_c\left(\hat{X}\cdot\phi^{-1}\right)\right\} \tag{322}$$
$$= \phi^e{}_{[a}\phi^f{}_{b]}\nabla_f\left\{\gamma^c{}_e\nabla_c\left(\hat{X}\cdot\phi^{-1}\right)\right\} \tag{323}$$
$$= \phi^e{}_{[a}\phi^f{}_{b]}\left\{\gamma^c{}_e\nabla_f\nabla_c\left(\hat{X}\cdot\phi^{-1}\right) + \left(\nabla_f\gamma^c{}_e\right)\nabla_c\left(\hat{X}\cdot\phi^{-1}\right)\right\} \tag{324}$$
$$= \phi^e{}_{[a}\phi^f{}_{b]}\left\{\left(\delta^c{}_e + n^c n_e\right)\nabla_f\nabla_c\left(\hat{X}\cdot\phi^{-1}\right) + \left(n^c\nabla_f n_e + n_e\nabla_f n^c\right)\nabla_c\left(\hat{X}\cdot\phi^{-1}\right)\right\} \tag{325}$$
$$= \phi^e{}_{[a}\phi^f{}_{b]}\left\{\nabla_f\nabla_e\left(\hat{X}\cdot\phi^{-1}\right) + \left(-A_e n_f + K_{ef}\right)n^c\nabla_c\left(\hat{X}\cdot\phi^{-1}\right)\right\} \tag{326}$$
$$= \phi^{[e}{}_{[a}\phi^{f]}{}_{b]}\left\{\nabla_f\nabla_e\left(\hat{X}\cdot\phi^{-1}\right) + K_{ef}n^c\nabla_c\left(\hat{X}\cdot\phi^{-1}\right)\right\} \tag{327}$$
$$= \phi^{[e}{}_{[a}\phi^{f]}{}_{b]}\left\{\nabla_{[f}\nabla_{e]}\left(\hat{X}\cdot\phi^{-1}\right) + K_{[ef]}n^c\nabla_c\left(\hat{X}\cdot\phi^{-1}\right)\right\} \tag{328}$$
$$= 0, \tag{329}$$

where $\hat{X}$ is a scalar in $\hat{\Sigma}$. We conclude that $\mathbb{D}$ is transformed to the Levi-Civita connection $\hat{\mathbb{D}}$ in $\hat{\Sigma}$.

The Lie derivative along W^a of spatial tensors has spatial part as

$$\mathcal{L}_W X = W^a\mathbb{D}_a X, \tag{330}$$
$$\perp\left(\mathcal{L}_W Y_a\right) = \perp\left(W^b\nabla_b Y_a + Y_b\nabla_a W^b\right) \tag{331}$$
$$= \perp\left(n_a W^b K^c{}_b Y_c + W^b\mathbb{D}_b Y_a - n_a Y_b \mathbb{T} W^b + Y_b\mathbb{D}_a W^b\right) \tag{332}$$
$$= W^b\mathbb{D}_b Y_a + Y_b\mathbb{D}_a W^b, \tag{333}$$
$$\perp\left(\mathcal{W}Z_{ab}\right) = \perp\left(W^c\nabla_c Z_{ab} + Z_{cb}\nabla_a W^c + Z_{ac}\nabla_b W^c\right) \tag{334}$$
$$= W^c\mathbb{D}_c Z_{ab} + Z_{cb}\mathbb{D}_a W^c + Z_{ac}\mathbb{D}_b W^c. \tag{335}$$

Because the spatial part of Lie derivative can be described by the connection $\mathbb{D}$, the Lie derivatives are transformed as

$$\mathcal{L}_W X \to \mathcal{L}_{\hat{W}} \hat{X}, \tag{336}$$

where W is a spatial vector, $\hat{W}$ is the transformation of W, X is a spatial tensor of any rank, and $\hat{X}$ is the transformation of X. The Lie derivative along $(\partial/\partial t)^a$ of rank (0,2) tensor Z_{ab} becomes

$$\mathcal{L}_{(\partial/\partial t)} Z_{ab} = \mathcal{L}_{(\partial/\partial t)} \left(Z_{\mu\nu} \left(dx^\mu\right)_a \left(dx^\nu\right)_b\right) \tag{337}$$

$$= \frac{\partial Z_{\mu\nu}}{\partial t} \left(dx^\mu\right)_a \left(dx^\nu\right)_b, \tag{338}$$

where $\{x^\mu\}$ is the adapted coordinate. It transform as

$$\mathcal{L}_{(\partial/\partial t)} Z_{ab} \to \phi^c{}_a \phi^d{}_b \frac{\partial Z_{\mu\nu}}{\partial t} \left(dx^\mu\right)_c \left(dx^\nu\right)_d = \frac{\partial Z_{ij}}{\partial t} \left(d\hat{x}^i\right)_a \left(d\hat{x}^j\right)_b. \tag{339}$$

because $\phi^b{}_a \left(dt\right)_b = 0$.

As a result, the system of equations transformed to $\hat{\Sigma}$ is written in the adapted coordinate system $\left\{\hat{x}^i\right\}$ on $\hat{\Sigma}$:

$$N^{-1} \frac{\partial K}{\partial t} = N^{-1} \mathcal{L}_N K + \mathbb{D}_i A^i - \frac{1}{3} K^2 - \breve{K}_{ij} \breve{K}^{ij} + A_i A^i - \frac{1}{2} \left(\rho + S\right), \tag{340}$$

$$N^{-1} \frac{\partial \breve{K}_{ij}}{\partial t} = N^{-1} \left(\mathcal{L}_N \breve{K}\right)_{ij} + \mathbb{D}_{<i} A_{j>} + \breve{K}_{k<i} \breve{K}_{j>}{}^k + A_{<i} A_{j>} + \frac{1}{2} \breve{S}_{ij} - \breve{E}_{ij} + \frac{2}{3} \gamma_{ij} \breve{K}^{kl} \breve{K}_{kl}, \tag{341}$$

$$\begin{aligned} N^{-1} \frac{\partial \breve{E}_{ij}}{\partial t} &= N^{-1} \left(\mathcal{L}_N \breve{E}\right)_{ij} + \epsilon_k{}^l{}_{<i} \mathbb{D}^k \breve{B}_{j>l} - \frac{1}{2} \mathbb{T} \breve{S}_{ij} - \frac{1}{2} \mathbb{D}_{<i} P_{j>} - A_{<i} P_{j>} - \frac{1}{6} \breve{K}_{ij} \left(3\rho + S\right) \\ &\quad + 2\epsilon_k{}^l{}_{<i} A^k \breve{B}_{j>l} - \frac{1}{3} K \left(\frac{1}{2} \breve{S} + \breve{E}\right)_{ij} + \breve{K}^k{}_{<i} \left(-\frac{1}{2} \breve{S} + 5\breve{E}\right)_{j>k} + \frac{2}{3} \gamma_{ij} \breve{K}^{kl} \breve{E}_{kl}, \end{aligned} \tag{342}$$

$$\begin{aligned} N^{-1} \frac{\partial \breve{B}_{ij}}{\partial t} &= N^{-1} \left(\mathcal{L}_N \breve{B}\right)_{ij} - \epsilon_k{}^l{}_{<i} \mathbb{D}^k \left(-\frac{1}{2} \breve{S} + \breve{E}\right)_{j>l} + \frac{1}{2} \epsilon_k{}^l{}_{<i} \breve{K}^k{}_{j>} P_l \\ &\quad + 5\breve{K}^k{}_{<i} \breve{B}_{j>k} - \frac{1}{3} K \breve{B}_{ij} - 2\epsilon_k{}^l{}_{<i} A^k \breve{E}_{j>l} + \frac{2}{3} \gamma_{ij} \breve{K}^{kl} \breve{B}_{kl}, \end{aligned} \tag{343}$$

$$N^{-1} \frac{\partial \rho}{\partial t} = N^{-1} \mathcal{L}_N \rho - \mathbb{D}^i P_i - 2A^i P_i - \frac{1}{3} K \left(\rho + S\right) - \breve{K}^{ij} \breve{S}_{ij} - \frac{2}{3} K\rho, \tag{344}$$

$$N^{-1} \frac{\partial P_i}{\partial t} = N^{-1} \left(\mathcal{L}_N\right) P_i - \frac{1}{3} \mathbb{D}_i S - \mathbb{D}^j \breve{S}_{ij} - K P_i - \frac{1}{3} A_i \left(3\rho + S\right) - A^j \breve{S}_{ij} \tag{345}$$

$$\frac{1}{2} P_i = \frac{1}{3} \mathbb{D}_i K - \frac{1}{2} \mathbb{D}^j \breve{K}_{ij}, \tag{346}$$

$$\breve{B}_{ij} = \epsilon_{kl<i} \mathbb{D}^k \breve{K}_{j>}{}^l, \tag{347}$$

$$0 = -\frac{1}{3} \mathbb{D}_i \rho + \mathbb{D}^j \left(\frac{1}{2} \breve{S} + \breve{E}\right)_{ij} + \frac{1}{3} K P_i - \epsilon^k{}_{ij} \breve{K}^{jl} \breve{B}_{kl} - \frac{1}{2} \breve{K}^j{}_i P_j, \tag{348}$$

$$0 = -\mathbb{D}^j \breve{B}_{ij} - \frac{1}{2} \epsilon^{jk}{}_i \mathbb{D}_j P_k - \epsilon^k{}_{ij} \breve{K}^{jl} \left(\frac{1}{2} \breve{S} + \breve{E}\right)_{kl}, \tag{349}$$

where we have omitted hat.

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