3+1 Formalism

2021 Summer School on Numerical Relativity and Gravitational Waves

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References

[1] George F R Ellis and Henk Van Elst. Cosmological models. In Marc Lachièze-Rey, editor, *Theoretical and observational cosmology*, number v. 541 in NATO science series, pages 1–116. Kluwer Academic, Dordrecht ; Boston, 1999.

[2] Eric Gourgoulhon. *3+1 Formalism in General Relativity*, volume 846. Springer Berlin Heidelberg, Berlin, Heidelberg, 2012. Series Title: Lecture Notes in Physics.

[3] Robert M Wald. *General Relativity*. University of Chicago Press, Chicago, 1984.

* Conventions

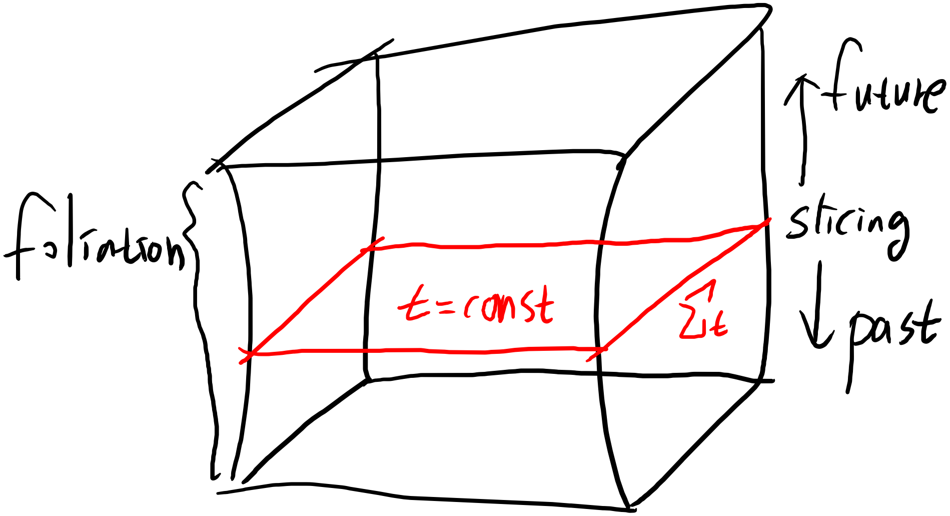
- unit
  - (natural unit) $c = 1$
  - (geometrized unit) $8\pi G = 1 \Rightarrow G_{ab} = T_{ab}$
- index
  - (abstract index) $a, b, c, \cdots$
  - (spacetime index) $\mu, \nu, \cdots = 0, 1, 2, 3$
  - (spatial index) $i, j, k, \cdots = 1, 2, 3$
- letter convention
  - (spacetime tensor) lower case (ex: r, c, t, f)
  - (spatial tensor) upper case (ex: A, K, S, P, E, B)

* Given Structure

(M, g) : globally hyperbolic spacetime

t : time function

Σ_t : level surface of t $\Rightarrow$ spacelike Cauchy surface



For a timelike vector v,

$$\begin{cases} v(t) > 0 : \text{future directed} \\ v(t) < 0 : \text{past directed} \end{cases}$$

* Normal Vector

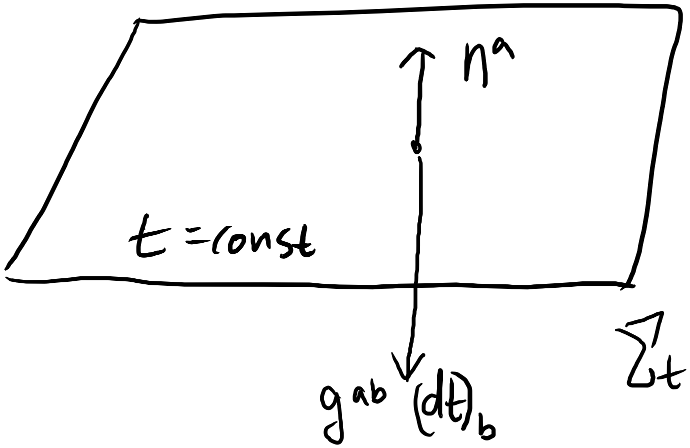
(gradient of t) $(dt)_a$ $\Rightarrow$ $g^{ab}(dt)_a(dt)_b < 0$ because Σ_t is spacelike

(normalization factor) $N \equiv \left(-g^{ab}(dt)_a(dt)_b\right)^{-1/2}$

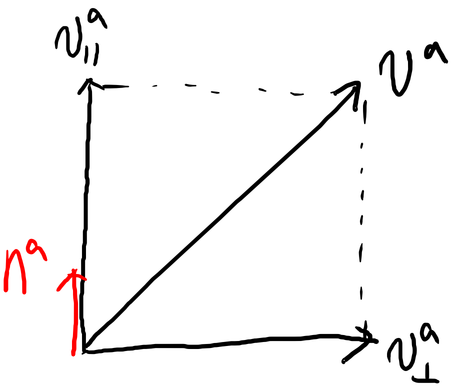
(normal vector) $n^a = -N g^{ab}(dt)_b$ $\Rightarrow$ $n(t) = n^a(dt)_a = -N g^{ab}(dt)_b(dt)_a$

$$= -N(-N^{-2}) = N^{-1} > 0$$

$$\Rightarrow \quad g_{ab}n^a n^b = -1$$



* Projections



$$v_{\parallel}{}^a = -n^a n_b v^b \ni n_a v_{\parallel}{}^a = -\overset{-1}{\boxed{n_a n^a}} n_b v^b = n_a v^a$$

$$v_{\perp}{}^a = v^a - v_{\parallel}{}^a = v^a + n^a n_b v^b = (\underbrace{\delta^a{}_b + n^a n_b}_{\gamma^a{}_b}) v^b$$

(orthogonal projection operator) $\gamma^a{}_b \equiv \delta^a{}_b + n^a n_b$

(orthogonality) $n_a \gamma^a{}_b = 0 \quad n^b \gamma^a{}_b = 0$ (#hw)
(trace) $\gamma^a{}_a = \delta^a{}_a + n^a n_a = 4 - 1 = 3$
(idempotence) $\gamma^a{}_c \gamma^c{}_b = \gamma^a{}_b$ (#hw)

* Orthogonal Decomposition

For a vector v,

$$v^a = A n^a + B^a \Rightarrow A = -n_a v^a , \quad B^a = \gamma^a{}_b v^b$$

For a linear form ω,

$$\omega_a = C n_a + D_a \Rightarrow C = -n^a \omega_a , \quad D_a = \gamma^b{}_a \omega_b$$

For a rank (0,2) tensor,

$$x_{ab} = n_a n_b A + n_a B_b + C_a n_b + D_{ab} , \begin{cases} A = n^a n^b x_{ab} \\ B_a = -n^c \gamma^d{}_a x_{cd} \\ C_a = -\gamma^c{}_a n^d x_{cd} \\ D_{ab} = \gamma^c{}_a \gamma^d{}_b x_{cd} \end{cases}$$

* Spatial Metric

$$\left.\begin{aligned} n^a n^b g_{ab} &= -1 \\ -n^c \gamma^d{}_a g_{cd} &= 0 \\ -\gamma^c{}_a n^d g_{cd} &= 0 \\ \gamma^c{}_a \gamma^d{}_b g_{cd} &= \gamma_{ab} \end{aligned}\right\} \quad g_{ab} = -n_a n_b + \boxed{\gamma_{ab}}$$

Spatial metric

For spatial vectors X, Y

$$\gamma_{ab} X^a Y^b = \gamma^c{}_a \gamma^d{}_b g_{cd} X^a Y^b = g_{ab} X^a Y^b$$

* Spatial Levi-Civitan tensor

(spatial L-C tensor) $\epsilon_{abc} = n^d \varepsilon_{dabc}$

$$\Rightarrow \epsilon_{abc} n^a = n^d n^a \varepsilon_{dabc} = 0$$

(properties) $\epsilon^{abc} \epsilon_{def} = 6 \gamma^{[a}{}_d \gamma^b{}_e \gamma^{c]}{}_f$

$\epsilon^{abe} \epsilon_{cde} = 2 \gamma^{[a}{}_c \gamma^{b]}{}_d$

$\epsilon^{acd} \epsilon_{bcd} = 2 \gamma^a{}_b$

$\epsilon^{abc} \epsilon_{abc} = 6$

* Irreducible Decomposition for Spatial Rank (0,2) Tensors

X_{ab}: rank (0,2) tensor

$X_{(ab)} = \frac{1}{2}(X_{ab} + X_{ba}) = \frac{1}{3}\gamma_{ab}X + \check{X}_{ab}$ $\Rightarrow$ $X = \gamma^{ab}X_{ab}$, $\check{X}_{ab} = (\gamma^{(c}{}_{a}\gamma^{d)}{}_{b} - \frac{1}{3}\gamma_{ab}\gamma^{cd})X_{cd}$

$X_{[ab]} = \frac{1}{2}(X_{ab} - X_{ba}) = \epsilon^{c}{}_{ab}X_{c}$ $\Rightarrow$ $X_{a} = \frac{1}{2}\epsilon^{bc}{}_{a}X_{bc}$

$\Rightarrow$ $X_{ab} = \frac{1}{3}\gamma_{ab}X + \check{X}_{ab} + \epsilon^{c}{}_{ab}X_{c}$

- number of components

X : 1

$\check{X}_{ab}$: 5

X_{a} : 3

X_{ab} : 9

* Orthogonal Decomposition of Symmetric and Antisymmetric Tensors

x_{ab} : Symmetric (0,2) tensor $\Rightarrow x_{ab} = n_a n_b A + 2 n_{(a} B_{b)} + \frac{1}{3} \gamma_{ab} C + \check{C}_{ab}$

A : 1
B_a : 3
C : 1
$\check{C}_{ab}$: 5
x_{ab} : 10

y_{ab} : antisymmetric (0,2) tensor $\Rightarrow y_{ab} = 2 n_{[a} D_{b]} + \epsilon^{c}{}_{ab} E_c$

D_a : 3
E_a : 3
y_{ab} : 6

* Example : Stress-energy Tensor

$$t_{ab} = n_a n_b \rho + 2 n_{(a} P_{b)} + \frac{1}{3} \gamma_{ab} S + \check{S}_{ab} \Rightarrow t = -\rho + S$$

$$\rho = n^a n^b t_{ab} \,, \quad P_a = -\gamma^c{}_a n^d t_{cd} \,, \quad S = \gamma^{ab} t_{ab} \,, \quad \check{S}_{ab} = \left(\gamma^{(c}{}_a \gamma^{d)}{}_b - \frac{1}{3} \gamma_{ab} \gamma^{cd}\right) t_{cd}$$

* Example : Field Strength Tensor

$$f_{ab} = 2 n_{[a} E_{b]} + \epsilon^c{}_{ab} B_c$$

$$E_a = f_{ab} n^b \,, \quad B_a = \frac{1}{2} \epsilon^{bc}{}_a f_{bc}$$

* Ricci Tensor and Scalar

(Einstein equation) $r_{ab} = t_{ab} - \frac{1}{2} g_{ab} t$

$$= n_a n_b \rho + 2n_{(a}P_{b)} + \frac{1}{3}\gamma_{ab} S + \breve{S}_{ab} - \frac{1}{2}(-n_a n_b + \gamma_{ab})(-\rho + S)$$

$$= \frac{1}{2}(S+\rho) n_a n_b + 2n_{(a}P_{b)} + \frac{1}{6}\gamma_{ab}(3\rho - S) + \breve{S}_{ab}$$ (#hw)

(traceless Ricci tensor) $\bar{r}_{ab} = r_{ab} - \frac{1}{4} g_{ab} r$

$$= \frac{1}{4} n_a n_b (3\rho + S) + 2n_{(a}P_{b)} + \frac{1}{12}\gamma_{ab}(3\rho + S) + \breve{S}_{ab}$$ (#hw)

* Ricci Decomposition

For a rank (0,4) tensor χ,

$$\chi_{abcd} = a\, g_{ab} g_{cd} + b\, g_{ac} g_{bd} + c\, g_{ad} g_{bc} + g_{ab} d_{cd} + g_{ac} e_{bd} + g_{ad} f_{bc} + h_{abcd}$$

where d, e, f, h are traceless.

$$\Rightarrow \chi^{[ab]}{}_{[cd]} = (b-c)\, \delta^{[a}{}_{[c} \delta^{b]}{}_{d]} + \delta^{[a}{}_{[c} \left(e^{b]}{}_{d]} - f^{b]}{}_{d]} \right) + h^{[ab]}{}_{[cd]}$$

For Riemann tensor r,

$$r^{ab}{}_{cd} = \frac{1}{6} \delta^{[a}{}_{[c} \delta^{b]}{}_{d]}\, r + 2 \delta^{[a}{}_{[c} \bar{r}^{b]}{}_{d]} + \underbrace{C^{ab}{}_{cd}}_{\text{Weyl tensor}}$$

* Orthogonal Decomposition of the Riemann Tensor

∘ Rank (2,2) Tensor of the Riemann Type

For a tensor $\chi^{ab}{}_{cd}$ such that $\chi^{(ab)}{}_{cd} = \chi^{ab}{}_{(cd)} = 0$

$$\chi^{ab}{}_{cd} = 2n^{[a}\left(2n_{[c}A^{b]}{}_{d]} + \epsilon^{|f|}{}_{cd}\, B^{b]}{}_{f}\right) \qquad \ni 36$$
$$+ \epsilon^{ab}{}_{e}\left(2n_{[c}C^{e}{}_{d]} + \epsilon^{f}{}_{cd}\, D^{e}{}_{f}\right)$$

$$A^{a}{}_{b} = n_c n^d \chi^{ac}{}_{bd} \qquad \ni 9$$
$$B^{a}{}_{b} = \frac{1}{2} n_c \epsilon^{de}{}_{b} \chi^{ac}{}_{de} \qquad \ni 9$$
$$C^{a}{}_{b} = \frac{1}{2} \epsilon^{a}{}_{de} n^c \chi^{de}{}_{bc} \qquad \ni 9$$
$$D^{a}{}_{b} = \frac{1}{4} \epsilon^{a}{}_{cd} \epsilon^{ef}{}_{b} \chi^{cd}{}_{ef} \qquad \ni 9$$

- First term

$$\delta^a{}_b = -n^a n_b + \gamma^a{}_b$$

$\cdot$

$\cdot$

- Weyl tensor

$$C^{ab}{}_{cd} = 4 n^{[a} n_{[c} \underset{\substack{\uparrow \\ \text{electric} \\ \text{part}}}{\check{E}^{b]}{}_{d]}} + 2 n^{[a} \epsilon^{|f|}{}_{cd} \underset{\substack{\uparrow \\ \text{magnetic} \\ \text{part}}}{\check{B}^{b]}{}_f} + 2 \epsilon^{ab}{}_e n_{[c} \overset{\check{B}^e{}_d}{\overset{/\!/}{C^e{}_{d]}}} + \epsilon^{ab}{}_e \epsilon^f{}_{cd} \overset{-\check{E}^e{}_f}{\overset{/\!/}{D^e{}_f}}$$

$$\begin{cases} C^{ab}{}_{cb} = 0 \\ r^a{}_{[bcd]} = 0 \quad \text{(the first Bianchi identity)} \end{cases}$$

- Riemann Tensor

$$
\begin{aligned}
r^{ab}{}_{cd} = {} & 4 h^{[a} n_{[c} \left\{ \frac{1}{6} \gamma^{b]}{}_{d]} (\rho + S) - \frac{1}{2} \breve{S}^{b]}{}_{d]} + \breve{E}^{b]}{}_{d]} \right\} \\
& + 2 h^{[a} \epsilon^{|f|}{}_{cd} \left(\breve{B}^{b]}{}_{f} + \frac{1}{2} \epsilon^{b]}{}_{fg} \underline{P}^{g} \right) \\
& + 2 \epsilon^{ab}{}_{e} n_{[c} \left(\breve{B}^{e}{}_{d]} - \frac{1}{2} \epsilon^{e}{}_{d]g} \underline{P}^{g} \right) \\
& + \epsilon^{ab}{}_{e} \epsilon^{f}{}_{cd} \left\{ \frac{1}{3} \gamma^{e}{}_{f} \rho - \frac{1}{2} \breve{S}^{e}{}_{f} - \breve{E}^{e}{}_{f} \right\}
\end{aligned}
$$

- \# of components

$r^{ab}{}_{cd}$	:	20	
$C^{ab}{}_{cd}$	:	10	
$\breve{E}_{ab}$	:	5	
$\breve{B}_{ab}$	:	5	
ρ	:	1	$\}$ 10
S	:	1	
$\underline{P}_a$	:	3	
$\breve{S}_{ab}$	:	5	

Covariant Derivatives

* Acceleration & Shape operator

observing,

$$n_a \nabla_b n^a = \nabla_b (n_a n^a) - n^a \nabla_b n_a$$

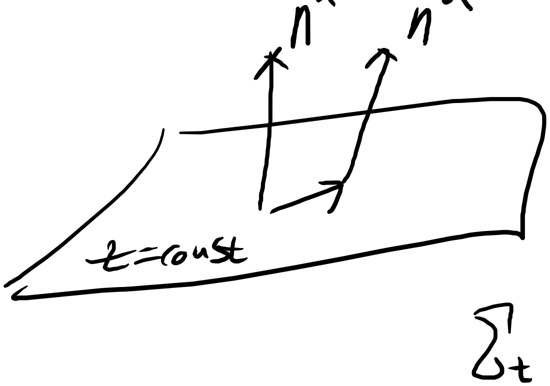
$$= \quad \text{''} \quad - n_a \nabla_b n^a$$

$$= \frac{1}{2} \nabla_b (\underset{\overset{\shortparallel}{-1}}{n_a n^a})$$

$$= 0$$

$b \backslash a$	T	S
T	0	-A
S	0	K

$$\ni \quad \nabla_b n^a = -A^a n_b + K^a{}_b$$



$$\ni \begin{cases} A^a = n^b \nabla_b n^a & \ni \text{ acceleration} \\ K^a{}_b = \gamma^c{}_b \nabla_c n^a & \ni \text{ shape operator} \end{cases}$$

Weingarten map

extrinsic curvature

second fundamental form

$$\ni \underbrace{\begin{cases} A_a = \gamma^b{}_a \nabla_b \ln N \\ K_{[ab]} = 0 \end{cases}}_{(\#hw)}$$

* Temporal & Spatial Derivatives

For a tensor X of any rank,

$$\mathbb{T} X \equiv \perp (n^a \nabla_a X)$$
$$\mathbb{D}_a X \equiv \perp (\nabla_a X)$$

where $\perp$ is the orthogonal projection operator.

$$\ni \begin{cases} \mathbb{T} \gamma_{ab} = 0 \\ \mathbb{D}_c \gamma_{ab} = 0 \\ \mathbb{T} \epsilon_{abc} = 0 \\ \mathbb{D}_d \epsilon_{abc} = 0 \end{cases}$$

(torsion free) $\mathbb{D}_{[a} \mathbb{D}_{b]} X = 0$ where X is a scalar.

* Orthogonal Decomposition of Covariant Derivatives

For a scalar X,

$$\nabla_a X = \delta^b{}_a \nabla_b X = (-n^b n_a + \gamma^b{}_a) \nabla_b X = -n_a \mathbb{T} X + \mathbb{D}_a X$$

For a linear form Y, (spatial)

$$\nabla_b Y_a = -n_a n_b A^c Y_c + n_a K^c{}_b Y_c - n_b \mathbb{T} Y_a + \mathbb{D}_b Y$$

For a rank (0,2) tensor Z, (spatial)

$$\nabla_c Z_{ab} = -n_a Z_{db} A^d n_c + n_a Z_{db} K^d{}_c - Z_{ad} A^d n_b n_c + Z_{ad} n_b K^d{}_c - n_c \mathbb{T} Z_{ab} + \mathbb{D}_c Z_{ab}.$$

$$\swarrow \quad \nabla_c n^a = -A^a n_a + K^a{}_c$$

(Ricci Identity) $r^a{}_{dbc} n^d = 2 \nabla_{[b} \nabla_{c]} n^a$

(second Bianchi identity) $\nabla_{[e} r^a{}_{bcd]} = 0$

* Orthogonal Decomposition of the Ricci identity

$$\nabla_c \nabla_b n^a = \nabla_c(-A^a n_b + K^a{}_b) = -n_b \nabla_c A^a - A^a(-A_b n_c + K_{bc}) + \nabla_c K^a{}_b$$

$$= -n_b(-n^a n_c A^d A_d + n^a K^d{}_c A_d - n_c \mathbb{T} A^a + \mathbb{D}_c A^a) + A^a A_b n_c - A^a K_{bc}$$

$$- n^a K_{db} A^d n_c + n^a K_{db} K^d{}_c - K_{ad} A^d n_b n_c + K_{ad} n_b K^d{}_c - n_c \mathbb{T} K_{ab} + \mathbb{D}_c K_{ab}$$

$$= n^a n_b n_c A^d A_d - n^a n_b K^d{}_c A_d - n^a n_c K_{db} A^d + n^a K_{db} K^d{}_c + n_b n_c \mathbb{T} A^a - n_b n_c K^a{}_d A^d - n_b \mathbb{D}_c A^a + n_b K^a{}_d K^d{}_c$$

$$+ n_c A^a A_b - n_c \mathbb{T} K^a{}_b - A^a K_{bc} + \mathbb{D}_c K^a{}_b. \qquad ($$

$$2\nabla_{[b}\nabla_{c]} n^a = 2n_{[b}\left(-\mathbb{T}K^a{}_{c]} + \mathbb{D}_{c]}A^a - K^a{}_{|d|}K^d{}_{c]} + A^a A_{c]}\right) + \epsilon^d{}_{bc}\left(\epsilon^{ef}{}_d \mathbb{D}_e K^a{}_f\right)$$

$$= 2 n_{[b} A^a{}_{c]} + \epsilon^d{}_{bc} B^a{}_d$$

For a rank (0,2) spatial tensor X,

$$X_{<ab>} = (\gamma^{(c}{}_a \gamma^{d)}{}_a - \frac{1}{3}\gamma^{cd}\gamma_{ab}) X_{cd}$$

evolution

$$\mathfrak{A} = -\mathbb{T}K + \mathbb{D}_a A^a - \frac{1}{3}K^2 - \breve{K}_{ab}\breve{K}^{ab} + A_a A^a, \qquad = \frac{1}{2}(\rho + S)$$

$$\breve{\mathfrak{A}}_{ab} = -\mathbb{T}\breve{K}_{ab} + \mathbb{D}_{<a}A_{b>} - \frac{2}{3}K\breve{K}_{ab} - \breve{K}_{c<a}\breve{K}_{b>}{}^c + A_{<a}A_{b>}, \quad = -\frac{1}{2}\breve{S}_{ab} + \breve{E}_{ab}$$

$$\mathfrak{A}^a = -\frac{1}{2}\epsilon^{abc}\mathbb{D}_b A_c, = -\frac{1}{2}\epsilon^{abc}\mathbb{D}_{[b}\mathbb{D}_{c]}\ln N = 0 \qquad = 0$$

$$\mathfrak{B} = 0, \qquad = 0$$

constraint

$$\breve{\mathfrak{B}}_{ab} = \epsilon_{cd<a}\mathbb{D}^c \breve{K}_{b>}{}^d, \qquad = \breve{B}_{ab}$$

$$\mathfrak{B}^a = \frac{1}{3}\mathbb{D}^a K - \frac{1}{2}\mathbb{D}_b \breve{K}^{ab}. \qquad = \frac{1}{2}\underline{p}^a$$

$$r^a{}_{dbc} n^d =$$

* Orthogonal Decomposition of the Second Bianchi Identity

$$0 = \nabla_{[e} r^{a}{}_{|b|cd]} \longleftrightarrow 0 = \nabla^{d} \left(\tfrac{1}{2} \varepsilon^{gh}{}_{cd} \, r^{ab}{}_{gh} \right)$$

$$r^{ab}{}_{cd} = 4 n^{[a} n_{[c} A^{b]}{}_{d]} + 2 n^{[a} \epsilon^{|f|}{}_{cd} B^{b]}{}_{f} + 2 \epsilon^{ab}{}_{e} n_{[c} C^{e}{}_{d]} + \epsilon^{ab}{}_{e} \epsilon^{f}{}_{cd} D^{e}{}_{f}$$

$$\tfrac{1}{2} \varepsilon^{gh}{}_{cd} \, r^{ab}{}_{gh} = \text{''} \left(- B^{b]}{}_{d]} \right) + \text{''} \left(A^{b]}{}_{f} \right) + \text{''} \left(- D^{e}{}_{d]} \right) + \text{''} \left(C^{e}{}_{f} \right)$$

$$0 = \nabla^{d} \left(\tfrac{1}{2} \varepsilon^{gh}{}_{cd} \, r^{ab}{}_{gh} \right) = 2n^{[a|} \left\{ n_c \left(-\nabla^d \mathfrak{B}^{|b]}{}_d + \epsilon^{|b]}{}_{de} K^{df} \mathfrak{D}^e{}_f \right) \right.$$
$$- K^d{}_c \mathfrak{B}^{|b]}{}_d + K \mathfrak{B}^{|b]}{}_c + n_d \nabla^d \mathfrak{B}^{|b]}{}_c + A^e \epsilon_e{}^f{}_c \mathfrak{A}^{|b]}{}_f + \epsilon^f{}_{cd} \nabla^d \mathfrak{A}^{|b]}{}_f + \epsilon^{|b]}{}_{ef} A^f \mathfrak{D}^e{}_c + \left. K^{dg} \epsilon_g{}^{|b]}{}_e \epsilon^f{}_{cd} \mathfrak{C}^e{}_f \right\}$$
$$+ \epsilon^{ab}{}_d \left\{ n_c \left(-\epsilon^d{}_{ef} K^{ge} \mathfrak{B}^f{}_g - \nabla^e \mathfrak{D}^d{}_e \right) \right.$$
$$\left. + \epsilon^d{}_{ef} \left(A^e \mathfrak{B}^f{}_c + K^{ge} \epsilon^h{}_{cg} \mathfrak{A}^f{}_h \right) - K^e{}_c \mathfrak{D}^d{}_e + K \mathfrak{D}^d{}_c + n_e \nabla^e \mathfrak{D}^d{}_c + A^g \epsilon_g{}^f{}_c \mathfrak{C}^d{}_f + \epsilon^f{}_{ce} \nabla^e \mathfrak{C}^d{}_f \right\}.$$

$$= 2 n^{[a} \left(n_c \mathcal{E}^{b]} + \mathcal{F}^{b]}{}_{c} \right) + \epsilon^{ab}{}_{d} \left(n_c \mathcal{G}^{d} + \mathcal{H}^{d}{}_{c} \right)$$

$$0 = \mathcal{E}_a , \quad \mathcal{F} = 0 , \quad \check{\mathcal{F}}_{ab} = 0 , \quad \mathcal{F}_a = 0 , \quad 0 = \mathcal{G}_a , \quad \mathcal{H} = 0 , \quad \check{\mathcal{H}}_{ab} = 0 , \quad \mathcal{H}_a = 0$$

* System of Equation

- evolution equation

$$\frac{1}{2}(\rho+S) = -\mathbb{T}K + \mathbb{D}_a A^a - \frac{1}{3}K^2 - \breve{K}_{ab}\breve{K}^{ab} + A_a A^a,$$

$$-\frac{1}{2}\breve{S}_{ab} + \breve{E}_{ab} = -\mathbb{T}\breve{K}_{ab} + \mathbb{D}_{<a}A_{b>} - \frac{2}{3}K\breve{K}_{ab} - \breve{K}_{c<a}\breve{K}_{b>}{}^{c} + A_{<a}A_{b>},$$

$$0 = 2\epsilon_{cd<a}A^c\breve{B}^d{}_{b>} - A_{<a}P_{b>} - K\left(\frac{1}{6}\breve{S} + \breve{E}\right)_{ab} - \frac{1}{6}\breve{K}_{ab}(3\rho+S) + \breve{K}_{c<a}\left(-\frac{1}{2}\breve{S} + 3\breve{E}\right)_{b>}^{\quad c}$$
$$- \frac{1}{2}\mathbb{T}\breve{S}_{ab} - \mathbb{T}\breve{E}_{ab} + \epsilon_{cd<a}\mathbb{D}^c\breve{B}^d{}_{b>} - \frac{1}{2}\mathbb{D}_{<a}P_{b>},$$

$$0 = -3\breve{K}^c{}_{<a}\breve{B}_{b>c} - \frac{1}{2}\epsilon_{cd<a}\breve{K}^c{}_{b>}P^d + K\breve{B}_{ab} + \mathbb{T}\breve{B}_{ab} + 2\epsilon_{cd<a}A^c\breve{E}^d{}_{b>} + \epsilon_{cd<a}\mathbb{D}^c\left(-\frac{1}{2}\breve{S} + \breve{E}\right)_{b>}^{\quad d},$$

$$0 = 2A^aP_a + \frac{1}{3}K(\rho+S) + \breve{K}_{ab}\breve{S}^{ab} + \frac{2}{3}K\rho + \mathbb{T}\rho + \mathbb{D}^aP_a,$$

$$0 = \epsilon_{abc}\breve{K}^{bd}\breve{B}^c{}_d + \frac{3}{2}\breve{K}_{ab}P^b + KP_a + \mathbb{T}P_a + \frac{1}{3}A_a(3\rho+S) + \frac{1}{3}\mathbb{D}_a(\rho+S) - \mathbb{D}^b\left(-\frac{1}{2}\breve{S} + \breve{E}\right)_{ab} + A^b\breve{S}_{ab}.$$

↳ $$0 = \breve{K}^b{}_aP^b + \frac{4}{3}KP_a + \mathbb{T}P_a + \frac{1}{3}A_a(3\rho+S) + \frac{1}{3}\mathbb{D}_aS + \mathbb{D}^b\breve{S}_{ab} + A^b\breve{S}_{ab}.$$

- constraint equation

$$\frac{1}{2}P^a = \frac{1}{3}\mathbb{D}^aK - \frac{1}{2}\mathbb{D}_b\breve{K}^{ab},$$

$$\breve{B}_{ab} = \epsilon_{cd<a}\mathbb{D}^c\breve{K}_{b>}{}^{d},$$

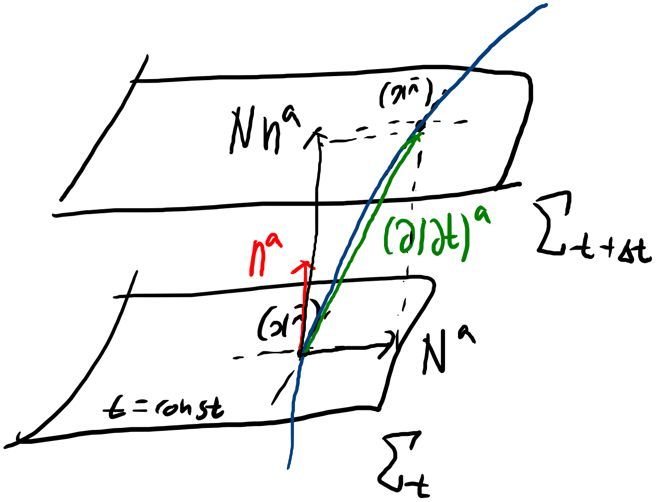
$$0 = \frac{1}{3}KP_a - \epsilon^c{}_{ab}\breve{K}^{bd}\breve{B}_{cd} - \frac{1}{2}\breve{K}^b{}_aP_b - \frac{1}{3}\mathbb{D}^a\rho + \mathbb{D}^b\left(\frac{1}{2}\breve{S} + \breve{E}\right)_{ab},$$

$$0 = -\mathbb{D}_b\breve{B}^{ab} - \frac{1}{2}\epsilon^{abc}\mathbb{D}_bP_c - \epsilon^a{}_{bc}\breve{K}^{bd}\left(\frac{1}{2}\breve{S} + \breve{E}\right)_d^{\ c}.$$

Application to Numerical Relativity

* Adapted Coordinate

$\{x^\mu : \mu = 0, 1, 2, 3\}$: adapted coordinate such that $x^0 = t$



$(\partial/\partial t)^a$: coordinate basis of t

$$(\partial/\partial t)^a = N n^a + N^a$$

[N : lapse function ∋ time slicing
N^a : shift vector ∋ spatial coordinate

For a scalar X,

$$\mathcal{L}_{(\partial/\partial t)} X = \mathcal{L}_N X + N n^a \nabla_a X$$
$$= \mathcal{L}_N X + N \Pi X$$

For a Y_a,

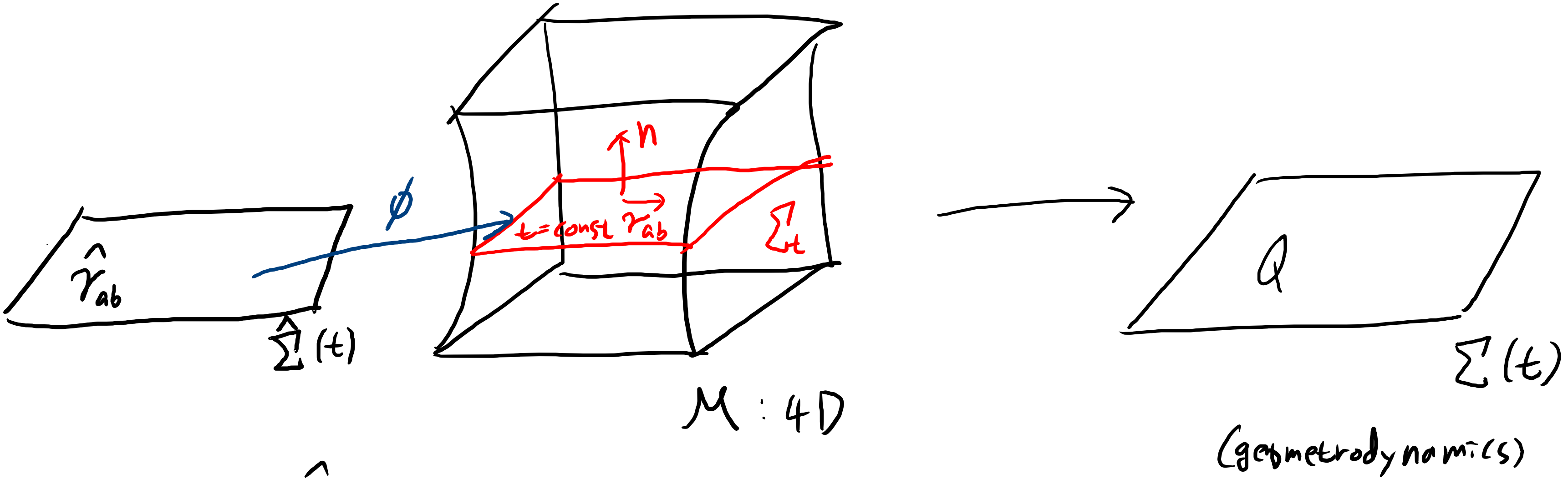
$$\mathcal{L}_{(\partial/\partial t)} Y_a = \mathcal{L}_N Y_a + N(\Pi Y_a + Y_b K^b{}_a)$$

For a Z_{ab},

$$\mathcal{L}_{(\partial/\partial t)} Z_{ab} = \mathcal{L}_N Z_{ab} + N(\Pi Z_{ab} + Z_{cb} K^c{}_a + Z_{ac} K^c{}_b)$$

$$
\begin{aligned}
N^{-1}\mathcal{L}_{(\partial/\partial t)}K &= N^{-1}\mathcal{L}_N K + \mathbb{D}_a A^a - \frac{1}{3}K^2 - \breve{K}_{ab}\breve{K}^{ab} + A_a A^a - \frac{1}{2}\left(\rho + S\right), \\
N^{-1}\mathcal{L}_{(\partial/\partial t)}\breve{K}_{ab} &= N^{-1}\mathcal{L}_N \breve{K}_{ab} + \mathbb{D}_{<a}A_{b>} + \breve{K}_{c<a}\breve{K}_{b>}{}^{c} + A_{<a}A_{b>} + \frac{1}{2}\breve{S}_{ab} - \breve{E}_{ab} + \underline{\frac{2}{3}\gamma_{ab}\breve{K}^{cd}\breve{K}_{cd}}, \\
N^{-1}\mathcal{L}_{(\partial/\partial t)}\breve{E}_{ab} &= N^{-1}\mathcal{L}_N \breve{E}_{ab} + \epsilon_{cd<a}\mathbb{D}^c\breve{B}^{d}{}_{b>} - \frac{1}{2}\mathbb{T}\breve{S}_{ab} - \frac{1}{2}\mathbb{D}_{<a}P_{b>} - A_{<a}P_{b>} - \frac{1}{6}\breve{K}_{ab}\left(3\rho + S\right) \\
&\quad + 2\epsilon_{cd<a}A^c\breve{B}^{d}{}_{b>} - \frac{1}{3}K\left(\frac{1}{2}\breve{S} + \breve{E}\right)_{ab} + \breve{K}_{c<a}\left(-\frac{1}{2}\breve{S} + 5\breve{E}\right)_{b>}{}^{c} + \underline{\frac{2}{3}\gamma_{ab}\breve{K}^{cd}\breve{E}_{cd}}, \\
N^{-1}\mathcal{L}_{(\partial/\partial t)}\breve{B}_{ab} &= N^{-1}\mathcal{L}_N \breve{B}_{ab} - \epsilon_{cd<a}\mathbb{D}^c\left(-\frac{1}{2}\breve{S} + \breve{E}\right)^{d}{}_{b>} + \frac{1}{2}\epsilon_{cd<a}\breve{K}^{c}{}_{b>}P^d \\
&\quad + 5\breve{K}^{c}{}_{<a}\breve{B}_{b>c} - \frac{1}{3}K\breve{B}_{ab} - 2\epsilon_{cd<a}A^c\breve{E}^{d}{}_{b>} + \underline{\frac{2}{3}\gamma_{ab}\breve{K}^{cd}\breve{B}_{cd}}, \\
N^{-1}\mathcal{L}_{(\partial/\partial t)}\rho &= N^{-1}\mathcal{L}_N \rho - \mathbb{D}^a P_a - 2A^a P_a - \frac{1}{3}K\left(\rho + S\right) - \breve{K}_{ab}\breve{S}^{ab} - \frac{2}{3}K\rho, \\
N^{-1}\mathcal{L}_{(\partial/\partial t)}P_a &= N^{-1}\mathcal{L}_N P_a - KP_a - \frac{1}{3}A_a\left(3\rho + S\right) - \frac{1}{3}\mathbb{D}_a S - \mathbb{D}^b\breve{S}_{ab} - A^b\breve{S}_{ab} \\
\frac{1}{2}P^a &= \frac{1}{3}\mathbb{D}^a K - \frac{1}{2}\mathbb{D}_b\breve{K}^{ab}, \\
\breve{B}_{ab} &= \epsilon_{cd<a}\mathbb{D}^c\breve{K}_{b>}{}^{d}, \\
0 &= \frac{1}{3}KP^a - \epsilon^{a}{}_{bc}\breve{K}^{bd}\breve{B}^{c}{}_{d} - \frac{1}{2}\breve{K}^{ab}P_b - \frac{1}{3}\mathbb{D}^a\rho + \mathbb{D}_b\left(\frac{1}{2}\breve{S} + \breve{E}\right)^{ab}, \\
0 &= -\mathbb{D}_b\breve{B}^{ab} - \frac{1}{2}\epsilon^{abc}\mathbb{D}_b P_c - \epsilon^{a}{}_{bc}\breve{K}^{bd}\left(\frac{1}{2}\breve{S} + \breve{E}\right)^{c}{}_{d}.
\end{aligned}
$$

* Geometrodynamics



$\gamma_{ab} \to \hat{\gamma}_{ab}$

$\gamma^{a}{}_{b} \to \hat{\gamma}^{a}{}_{b}$

$\gamma_{ab} \to \hat{\gamma}_{ab}$

$\mathbb{D} \to \hat{\mathbb{D}}$

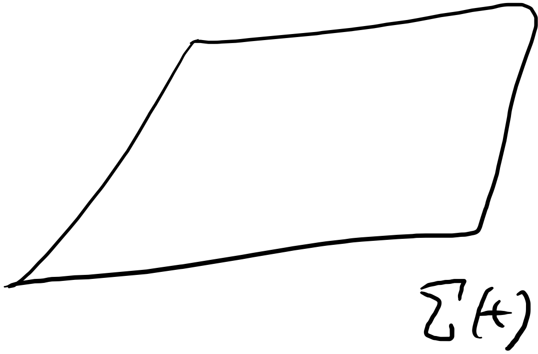
$\mathbb{D}_c \gamma_{ab} = 0 \to \hat{\mathbb{D}}_c \hat{\gamma}_{ab} = 0$

$\mathbb{D}_{[a}\mathbb{D}_{b]} f \to \hat{\mathbb{D}}_{[a}\hat{\mathbb{D}}_{b]}\hat{f} = 0$

$\mathcal{L}_{w} X \to \mathcal{L}_{\hat{w}} \hat{X}$

$\mathcal{L}_{(\partial/\partial t)} \tau_{ab} \to \frac{\partial \tau_{\hat{i}\hat{j}}}{\partial t} (d\hat{x}^{\hat{i}})_{\hat{a}} (d\hat{x}^{j})$

$$
\begin{aligned}
N^{-1}\frac{\partial K}{\partial t} &= N^{-1}\mathcal{L}_N K + \mathbb{D}_i A^i - \frac{1}{3}K^2 - \breve{K}_{ij}\breve{K}^{ij} + A_i A^i - \frac{1}{2}\left(\rho + S\right),\\
N^{-1}\frac{\partial \breve{K}_{ij}}{\partial t} &= N^{-1}\left(\mathcal{L}_N \breve{K}\right)_{ij} + \mathbb{D}_{<i}A_{j>} + \breve{K}_{k<i}\breve{K}_{j>}{}^{k} + A_{<i}A_{j>} + \frac{1}{2}\breve{S}_{ij} - \breve{E}_{ij} + \frac{2}{3}\gamma_{ij}\breve{K}^{kl}\breve{K}_{kl},\\
N^{-1}\frac{\partial \breve{E}_{ij}}{\partial t} &= N^{-1}\left(\mathcal{L}_N \breve{E}\right)_{ij} + \epsilon_k{}^{l}{}_{<i}\mathbb{D}^k\breve{B}_{j>l} - \frac{1}{2}\mathbb{T}\breve{S}_{ij} - \frac{1}{2}\mathbb{D}_{<i}P_{j>} - A_{<i}P_{j>} - \frac{1}{6}\breve{K}_{ij}\left(3\rho + S\right)\\
&\quad + 2\epsilon_k{}^{l}{}_{<i}A^k\breve{B}_{j>l} - \frac{1}{3}K\left(\frac{1}{2}\breve{S} + \breve{E}\right)_{ij} + \breve{K}^k{}_{<i}\left(-\frac{1}{2}\breve{S} + 5\breve{E}\right)_{j>k} + \frac{2}{3}\gamma_{ij}\breve{K}^{kl}\breve{E}_{kl},\\
N^{-1}\frac{\partial \breve{B}_{ij}}{\partial t} &= N^{-1}\left(\mathcal{L}_N \breve{B}\right)_{ij} - \epsilon_k{}^{l}{}_{<i}\mathbb{D}^k\left(-\frac{1}{2}\breve{S} + \breve{E}\right)_{j>l} + \frac{1}{2}\epsilon_k{}^{l}{}_{<i}\breve{K}^k{}_{j>}P_l\\
&\quad + 5\breve{K}^k{}_{<i}\breve{B}_{j>k} - \frac{1}{3}K\breve{B}_{ij} - 2\epsilon_k{}^{l}{}_{<i}A^k\breve{E}_{j>l} + \frac{2}{3}\gamma_{ij}\breve{K}^{kl}\breve{B}_{kl},\\
N^{-1}\frac{\partial \rho}{\partial t} &= N^{-1}\mathcal{L}_N\rho - \mathbb{D}^iP_i - 2A^iP_i - \frac{1}{3}K\left(\rho + S\right) - \breve{K}^{ij}\breve{S}_{ij} - \frac{2}{3}K\rho,\\
N^{-1}\frac{\partial P_i}{\partial t} &= N^{-1}\left(\mathcal{L}_N\right)P_i - \frac{1}{3}\mathbb{D}_iS - \mathbb{D}^j\breve{S}_{ij} - KP_i - \frac{1}{3}A_i\left(3\rho + S\right) - A^j\breve{S}_{ij}\\
\frac{1}{2}P_i &= \frac{1}{3}\mathbb{D}_iK - \frac{1}{2}\mathbb{D}^j\breve{K}_{ij},\\
\breve{B}_{ij} &= \epsilon_{kl<i}\mathbb{D}^k\breve{K}_{j>}{}^{l},\\
0 &= -\frac{1}{3}\mathbb{D}_i\rho + \mathbb{D}^j\left(\frac{1}{2}\breve{S} + \breve{E}\right)_{ij} + \frac{1}{3}KP_i - \epsilon^k{}_{ij}\breve{K}^{jl}\breve{B}_{kl} - \frac{1}{2}\breve{K}^j{}_iP_j,\\
0 &= -\mathbb{D}^j\breve{B}_{ij} - \frac{1}{2}\epsilon^{jk}{}_i\mathbb{D}_jP_k - \epsilon^k{}_{ij}\breve{K}^{jl}\left(\frac{1}{2}\breve{S} + \breve{E}\right)_{kl},
\end{aligned}
$$



Where ^ is omitted.

$i, j, k, \cdots = 1, 2, 3$

⇒ Confomal transformation

⇒ initial data

⇒ time Slicing

⇒ spatial coordinate

⇒ evolution scheme