Chapter 1

Introduction to Hydrodynamics

In this section, we review the fundamental concepts and equations of fluid dynamics in the flat-space based on Newtonian physics by introducing elastostatics and hydrostatics. After that, we extend and compare the concepts and equations with those in the Minkowski/curved spacetime to understand the relativistic fluid dynamics. This note was made by referring to the following references.

- K. S. Thorne and R. D. Blandford, "Modern Classical Physics," Princeton University Press (Princeton University Press, 2017).
- J. L. Friedman and N. Stergioulas, "Rotating Relativistic Stars," Cambridge University Press (Cambridge University Press, 2013).

In this note, we use the natural unit $(c = 1)$. We write down three-dimensional vectors in bold font, however, to avoid confusion, we distinguish the three-dimensional metric tensor and the three gravitational acceleration vector by denoting them as **g** and $\vec{g}$, respectively.

$\boldsymbol{T},\ T^{\mu\nu},\ \boldsymbol{v},\ v^{\mu}:$ four-dimensional vector

$$\mathsf{T},\ T^{ij},\ \mathsf{v},\ v^i:\ \text{three-dimensional vector}$$
$$*\ \vec{g}:\ \text{three-dimensional gravitational acceleration vector} \tag{1.1}$$

1.1 Elastostatics

To understand elastostatics, we should understand what the strain tensor and the stress tensor are. Those two tensors are the generalization of the displacement and the force in Hooke's law.

$$\boxed{\mathsf{dF} = -k\mathsf{dx}} \quad \text{(F: restoring force on surroundings)}$$
$$\Updownarrow$$
$$\boxed{(\mathsf{T})\cdot d\mathbf{\Sigma} = \mathsf{dF} = (-\mathsf{Y}:\mathsf{S})\cdot d\mathbf{\Sigma}}$$

where $\mathsf{T} = T_{ij}$: stress tensor

$d\mathbf{\Sigma}$: directed area element

$\mathsf{Y} = Y_{ijkl}$: elastic modulus tensor

$\mathsf{S} = S_{ij}$: strain tensor

$$\mathsf{T}\cdot d\mathbf{\Sigma} = T_{ij}d\Sigma^j,\ \mathsf{Y}:\mathsf{S} = Y_{ijkl}S_{kl} \tag{1.2}$$

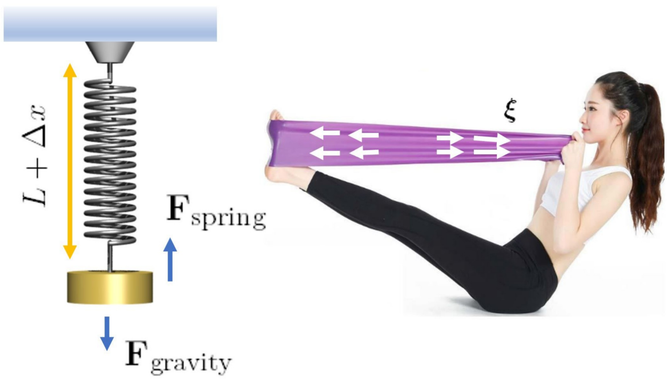

Figure 1.1: Generalization of the Hooke's law

The strain tensor is defined as the gradient of the displacement vector without considering any rotations. Since in elasticity theory we don't mind the specific orientation of an object, a central focus is on expansion and shear, and we name the expansion and shear part in the gradient of the displacement vector the strain:

Displacement vector: At each part, the point $\mathbf{x}$ is changed to $\boldsymbol{\xi}(\mathbf{x})$.

Then the gradient of the displacement vector is given by:

$\boldsymbol{\nabla\xi} = \underbrace{\frac{1}{3}\Theta\mathbf{g} + \boldsymbol{\Sigma}}_{\equiv \mathbf{S}} + \mathbf{R}$ (decomposition of the gradient of a displacement vector)

$$\begin{aligned}
\vec{\nabla}_i\xi_j &= \underbrace{\left[\frac{1}{2}(\vec{\nabla}_i\xi_j + \vec{\nabla}_j\xi_i)\right]}_{\text{(symmetric part)}} + \underbrace{\left[\frac{1}{2}(\vec{\nabla}_i\xi_j - \vec{\nabla}_j\xi_i)\right]}_{\text{(anti-symmetric part)}} \\
&= \overbrace{\left[\frac{1}{3}\underbrace{(\vec{\nabla}\cdot\boldsymbol{\xi})}_{\Theta} g_{ij}\right]}^{\text{(trace of symmetric part)}} + \overbrace{\left[\underbrace{\frac{1}{2}(\vec{\nabla}_i\xi_j + \vec{\nabla}_j\xi_i) - \frac{1}{3}(\vec{\nabla}\cdot\boldsymbol{\xi})g_{ij}}_{\equiv\Sigma_{ij}}\right]}^{\text{(trace-free of symmetric part)}} + \overbrace{\left[\underbrace{\frac{1}{2}(\vec{\nabla}_i\xi_j - \vec{\nabla}_j\xi_i)}_{\equiv R_{ij}}\right]}^{\text{(anti-symmetric part)}} \\
&\equiv \frac{1}{3}\Theta g_{ij} + \Sigma_{ij} + R_{ij} \qquad (1.3)
\end{aligned}$$

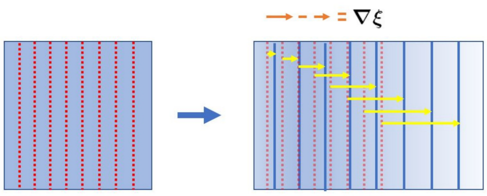

Figure 1.2: Displacement vector in elastostatics

The geometrical description of the displacement vector and its gradient:

(1) expansion scalar Θ: $\dfrac{\delta V}{V} \sim$ change of volume

(2) shear tensor: $\boldsymbol{\Sigma} \sim$ change of a shape

(3) rotation tensor: $\mathbf{R}$ (1.4)

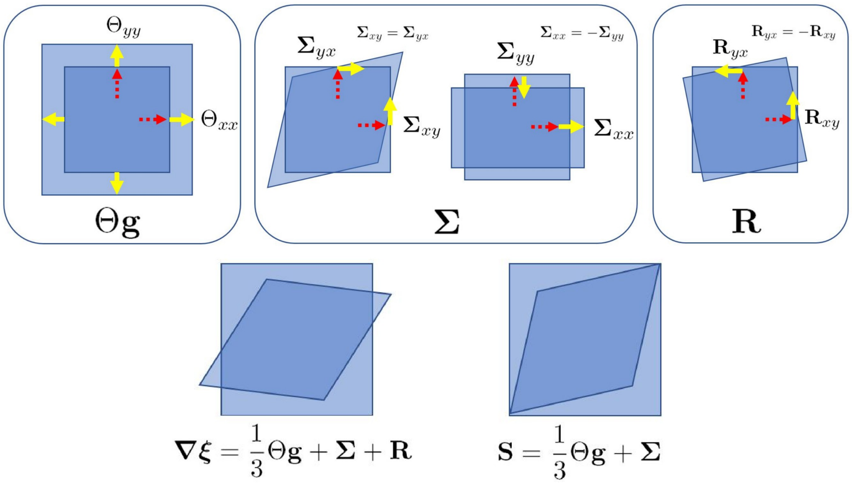

Figure 1.3: Each irreducible part of the gradient of the displacement vector corresponds to a geometrically specific change. The expansion and shear parts make up a strain tensor.

As we decompose the gradient of the displacement vector, we can repeat the same way on the stress tensor.

$$S_{ij} = \overbrace{\frac{1}{3}\Theta g_{ij}}^{\text{[trace]}} + \overbrace{\Sigma_{ij}}^{\begin{bmatrix}\text{trace-free}\\ \text{symmetric}\end{bmatrix}}$$

expansion ↑ ↑ shear

$$\boxed{\mathsf{d}\mathbf{F} = -k\mathsf{d}\mathbf{x}} \text{ (Hooke's law)}$$

$$\boxed{(\mathbf{T})\cdot d\mathbf{\Sigma} = \mathsf{d}\mathbf{F} = (-\mathbf{Y}:\mathbf{S})\cdot d\mathbf{\Sigma}}$$

$$\boxed{\mathbf{T} = -K\Theta\mathbf{g} - 2\mu\mathbf{\Sigma}}$$

pressure ↓ ↓ shear stress

$$T_{ij} = \underbrace{\left[\frac{1}{3}T_{ii}\right]}_{\text{(trace)}} + \underbrace{\left[\frac{1}{2}(T_{ij} - T_{ji}) - \frac{1}{3}T_{ii}\right]}_{\text{(trace-free symmetric)}}$$

$$\equiv P g_{ij} + T^{\text{shear}}_{ij} \tag{1.5}$$

The coefficients connecting the stress and the shear tensor parts are called

$$\boxed{P} \xleftarrow{K:\ \text{bulk modulus}} \boxed{\Theta}$$
$$\boxed{T^{\text{shear}}_{ij}} \xleftarrow{\mu:\ \text{shear modulus}} \boxed{\Sigma_{ij}}. \tag{1.6}$$

We can compute the total elastic force acting on the volume by considering the local condition:

$$\mathbf{F}_{\text{on } \mathcal{V} \text{ (by surroundings)}} = \int_{\partial \mathcal{V}} \mathbf{T} \cdot \overbrace{(-d\boldsymbol{\Sigma})}^{\text{(inward direction)}} = -\int_{\mathcal{V}} \underbrace{\boldsymbol{\nabla} \cdot \mathbf{T}}_{\equiv \mathbf{f}_{\text{elastic}}:\ \text{elastic force density (outward)}} dV$$

↳ $\boxed{\mathbf{f}_{\text{on } \mathcal{V}} \equiv -\boldsymbol{\nabla} \cdot \mathbf{T}}$ (elastic force density on a volume element) (1.7)

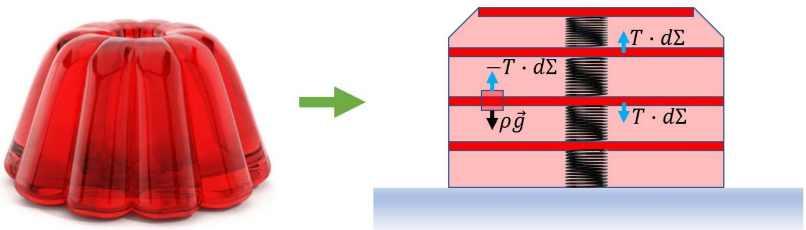

Figure 1.4: Elastostatic equilibrium

In elastostatic equilibrium, the net force density must be zero.

$$\mathbf{f}_{\text{on } \mathcal{V} \text{ (by surroundings)}} + \mathbf{f}_{\text{external}} = 0 = \mathbf{f}_{\text{net}}$$

↪ elastic force density on a volume element, $\mathbf{f}_{\text{on } \mathcal{V}} \equiv -\boldsymbol{\nabla} \cdot \mathbf{T}$

↪ in the presence of $\vec{g}$, $\mathbf{f}_{\text{external}} = \rho_0 \vec{g}$

↳ $\boxed{-\boldsymbol{\nabla} \cdot \mathbf{T} + \rho_0 \vec{g} = 0}$ (elastostatic equilibrium)

↪ Fluids at rest exert isotropic stresses: $\mathbf{T} = P\mathbf{g}$

↪ $-\boldsymbol{\nabla} \cdot \mathbf{T} = -\boldsymbol{\nabla} \cdot (P\mathbf{g}) = -\boldsymbol{\nabla} P$

↳ $\boxed{\boldsymbol{\nabla} P = \rho_0 \vec{g}}$ (elastostatic equilibrium in the presence of $\vec{g}$) (1.8)

When the elastic force on the volume element is not balanced with an external force, the net force makes the elastic element move. Then, we should consider its dynamics. This point will be seen again when we move from the hydrostatics to hydrodynamics.

$$\mathbf{f}_{\text{net}} = \mathbf{f}_{\text{on}\mathcal{V}} + \mathbf{f}_{\text{external}} \neq 0 \ \rightarrow \ (\text{dynamics}) \tag{1.9}$$

As examples of elastostatic equilibrium, in the case of the atmosphere of Earth, we expect the pressure is about 10^5Pa, and for Neutron stars, the magnitude of the stress at the base of a neutron star crust is about 10^{31}Pa.

When $P = P(z),\ \vec{g} = -g\hat{z}$,

$\boldsymbol{\nabla} P = \rho_0 \vec{g} \ \rightarrow \ P = \int_0^h \rho_0 g \mathrm{d}z \sim \rho_0 g h$ (for a constant mass density)

(1) Earth example:

$$\begin{cases} \text{air density:} & \rho_{0\text{air}} \sim 1\,\text{kg}\cdot\text{m}^{-3} \\ \text{gravity acceleration:} & g \sim 10\,\text{m}\,\text{s}^{-2} \\ \text{atmospheric scale height:} & h \sim 10\,\text{km} \end{cases}$$

↳ $P \sim \rho_0 g h \sim (1\,\text{kg}\,\text{m}^{-3})(10\,\text{m}\,\text{s}^{-2})(10\,\text{km}) \sim 10^5$ Pa

(2) Neutron star:

$$\begin{cases} \text{solid crust density:} & \rho_0 \sim 10^{16}\text{kg}\,\text{m}^{-3} \\ \text{gravity acceleration:} & g \sim \frac{GM}{R^2} \xrightarrow[R\,\sim\,10\text{km}]{M\,\sim\,2\,\times\,10^{30}\text{kg}} \frac{(6.67\text{m}^3\text{kg}^{-1}\text{s}^{-2})\times(2\times10^{30}\text{kg})}{(10\text{km})^2} \sim 10^{12}\text{m}\,\text{s}^{-2} \\ \text{crust thickness:} & h \sim 1\text{km} \end{cases}$$

$$\hookrightarrow P \sim \rho_0 g h \sim (10^{16}\text{kg}\,\text{m}^{-3})(10^{12}\text{m}\,\text{s}^{-2})(1\text{km}) \sim 10^{31}\text{Pa} \tag{1.10}$$

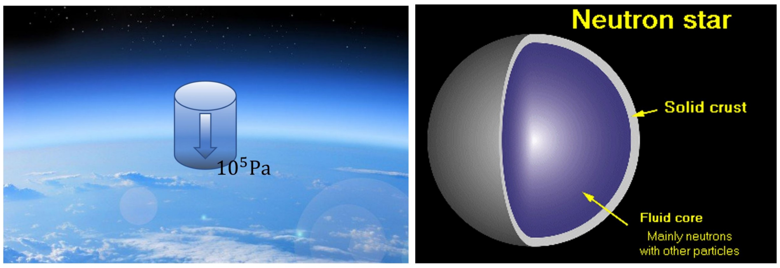

Figure 1.5: The atmosphere of the Earth and the solid crust of a neutron star

In this example, we considered the uniform gravitational field. When dealing with a fluid on the ground, we usually consider the gravitational acceleration constant everywhere. This is because the length scale of the laboratory on the Earth's surface is smaller than the Earth's radius.

When we express the equation for the elastostatic equilibrium in terms of the displacement vector, we get the Navier-Cauchy equation given by

$$
\begin{aligned}
&\boxed{\mathbf{f}_{\text{on }\mathcal{V}} = -\boldsymbol{\nabla}\cdot\mathbf{T}} \\
&= -\boldsymbol{\nabla}\cdot(-K\Theta\mathbf{g} - 2\mu\boldsymbol{\Sigma}) \\
&= \left(K + \frac{1}{3}\mu\right)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}) + \mu\nabla^2\boldsymbol{\xi} \\
\Lsh\ &\boxed{\mathbf{f}_{\text{on }\mathcal{V}} + \rho_0\vec{g} = \left(K + \frac{1}{3}\mu\right)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}) + \mu\nabla^2\boldsymbol{\xi} + \rho_0\vec{g} = 0} \\
&: \text{Navier-Cauchy equation}\,. \qquad (1.11)
\end{aligned}
$$

1.2 About fluids (density, pressure, velocity)

1.2.1 Solid and fluids

Imagine that there is water in the cup. The shape changes according to the container (a cup). When we put our hand in the cup, the water changes shape without any resistance. This is a normal behavior of a fluid. A fluid is a substance that is continuously deformed under applied shear stress or external force, that is, the shear modulus of the fluid is zero.

Solid: $\boxed{\mathbf{T} = -K\Theta\mathbf{g} - 2\mu\mathbf{\Sigma}}$

Fluid: $\mathbf{T} = \underbrace{-K\Theta\mathbf{g}}_{?} - \underbrace{\cancel{2\mu\mathbf{\Sigma}}}_{\mu\to 0:\ \text{change shape continuously}} \rightarrow$ It flows.

Fluid resists only relative rates of deformation

$$\left[\nabla\boldsymbol{\xi} = \underbrace{\frac{1}{3}\Theta\mathbf{g} + \quad}_{\equiv \mathbf{S}} + \mathbf{R}\right] \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}} \left[\nabla\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} = \nabla\mathbf{v} = \frac{1}{3}\theta\mathbf{g} + \boldsymbol{\sigma} + \mathbf{r}\right]$$

$$\boxed{\mathbf{T} = [\rho_0\mathbf{v}\otimes\mathbf{v} + P\mathbf{g}] - \zeta\theta\mathbf{g} - 2\eta\boldsymbol{\sigma}}$$

where ζ: bulk viscosity

η: shear viscosity

$\boldsymbol{\sigma}$: trace-free rate of shear tensor

$\mathbf{r}$: rate of rotation tensor (1.12)

1.2.2 Ideal fluid

The stress tensor of most fluids can be decomposed into an isotropic pressure and a viscous term linear in the rate of shear and compression. Under many conditions, the viscous stress can be neglected over most of the flow, and diffusive heat conductivity is negligible. The fluid is then called ideal or perfect.

Ideal(Perfect) fluid: no shear stress, no heat conductivity

→ isotropic pressure in the local rest frame of the fluid (1.13)

In this case, we can observe the macroscopic nature of a fluid as follows:

- **Fluid velocity:** The macroscopic continuum approximation is valid.

  molecular mean free paths $\ll$ macroscopic length-scales

$$\boxed{d_{\text{M.F.P.}} \ll d_{\text{fluid scale}}}$$

↳ A small part of a fluid looks like moving with the same velocity.

mean local velocity $\mathbf{v}(\mathbf{x}, t)$ of the fluid's molecules: fluid's velocity

$$\boxed{\text{fluid velocity: } \mathbf{v}_{\text{mean for } dV}(\mathbf{x}, t)} \tag{1.14}$$

- **Fluid stress tensor:** We study the laws governing ideal fluids. In this case, the fluids do not oppose a steady shear strain.
  ↳ This is easy to understand on microscopic grounds,
  as there is no lattice to deform,
  and the molecular velocity distribution remains
  locally isotropic in the presence of a static shear.

$$\boxed{\text{fluid stress tensor: } \mathbf{T}^{\text{fluid}} = P\mathbf{g}} \tag{1.15}$$

1.2.3 Bulk modulus and adiabatic index

The laws of fluid mechanics are equally valid for liquids, gases, and plasmas. Liquids show different behavior from gases and plasmas, especially under compression.

$$\begin{cases} \text{Hooke's law} & : \quad -kx = F \\ \text{solid} & : \quad -K\Theta = -K\dfrac{\delta V}{V} = P \ (\Theta = 0 \to P = 0) \\ \text{liquid} & : \quad \boxed{-K\Theta = -K\dfrac{\delta V}{V} = \delta P} \ \left(= K\dfrac{\delta\rho_0}{\rho_0}, \ \ K = -V\dfrac{\text{d}P}{\text{d}V}\right) \\ \text{gas, plasma} & : \quad \boxed{-\Gamma\Theta = -\Gamma\dfrac{\delta V}{V} = \dfrac{\delta P}{P}} \ \left(= \Gamma\dfrac{\delta\rho_0}{\rho_0}, \ \ \Gamma = -\dfrac{V}{P}\dfrac{\text{d}P}{\text{d}V}\right) \end{cases}$$

$$(\hookrightarrow K = \Gamma P)$$

where K: bulk modulus,

Θ: expansion $\dfrac{\delta V}{V} = -\dfrac{\delta\rho_0}{\rho_0}$

$$\frac{\delta\rho_0}{\rho_0} = \frac{\delta(m/V)}{m/V} = \frac{-m/V^2}{m/V}\delta V = -\frac{\delta V}{V}$$

$\Gamma \equiv \dfrac{c_P}{c_V}$: adiabatic index for adiabatic process in an ideal gas

$$(c_P,\ c_V\text{: specific heats at constant pressure and volume}) \tag{1.16}$$

In the case of liquids, the molecules resist even a small compression, and this behavior results in large pressure changes. Since the fluid has a non-zero pressure for zero compression, the bulk modulus relation is given in the same form as the solid relation except for P to δP. By contrast, gases and plasmas are much less resistant to compression, and when pressure doubles, it leads to half the pressure.

$$\begin{aligned}
&\text{solid to liquid:} \ -K\Theta = -K\frac{\delta V}{V} = P \to \delta P \\
&\text{gas and plasma:} \ \ -\Gamma\Theta = -\Gamma\frac{\delta V}{V} = \frac{\delta P}{P} \\
&\text{(ex) } PV = nRT,\ \mathrm{d}T = 0 \ \rightarrow\ \delta(PV) = (\delta P)V + P(\delta V) = 0 \\
&\qquad\qquad\qquad\qquad \hookrightarrow \frac{\delta P}{P} = -\frac{\delta V}{V} \\
&\qquad\qquad\qquad\qquad \hookrightarrow \Gamma = 1 \text{ (isothermal process)}
\end{aligned} \tag{1.17}$$

For gases or plasmas, a specific adiabatic index indicates a different thermodynamic process.

1.2.4 Adiabatic index and polytropic process

The adiabatic index Γ is defined as the ratio between the specific heats at constant pressure and volume. For a polytropic process where a particular form of pressure and volume, Pv^k (v: specific volume), is a constant during a quasi-equilibrium process, the polytropic index k is the same with the adiabatic index for an isentropic process.

(1) **polytropic process:**

$$\boxed{Pv^k = \text{constant}} \begin{cases} k = 0 & \text{isobaric process} \\ k = 1 & \text{isothermal process of ideal gas} \\ k = \Gamma & \text{isentropic process of ideal gas} \\ k \to \infty & \text{isometric(isochoric) process} \end{cases}$$

(2) **a different form of a polytropic process:**

$$\frac{\delta P}{P} \xrightarrow{Pv^k=\text{const.}} \frac{\delta v^{-k}}{v^{-k}} = -k\frac{v^{-k-1}\delta v}{v^{-k}} = -k\frac{\delta v}{v}$$

$$\therefore \boxed{\frac{\delta P}{P} = -k\frac{\delta v}{v} = k\frac{\delta \rho_0}{\rho_0}}$$

$$\Rightarrow \boxed{P \propto \rho_0^k \equiv K\rho_0^k = K\rho_0^{1+1/n}} : \text{ (polytropic EOS)}$$

($K, k(n)$: polytropic constant, exponent(index))

(3) **isentropic process (adiabatic and reversible):**

$\Delta s = 0,\ \mathrm{d}u = c_v \mathrm{d}T,\ Pv = \overline{R}T,\ (\overline{R}$: specific gas constant)

$$\hookrightarrow \frac{\mathrm{d}T}{T} = -\left(\frac{\overline{R}}{c_v}\right)\frac{\mathrm{d}v}{v} \rightarrow \ln T = -\underbrace{\left(\frac{\overline{R}}{c_v}\right)}_{=\frac{c_P}{c_V}-1\equiv\Gamma-1}\ln v + C$$

$$\therefore \begin{cases} \left(\dfrac{T_2}{T_1}\right) = \left(\dfrac{v_1}{v_2}\right)^{\Gamma-1} = \left(\dfrac{P_2}{P_1}\right)^{\frac{\Gamma-1}{\Gamma}} \\ \left(\dfrac{P_2}{P_1}\right) = \left(\dfrac{v_1}{v_2}\right)^{\Gamma} \end{cases} \Rightarrow \boxed{Pv^{\Gamma} = \text{constant}} \qquad (1.18)$$

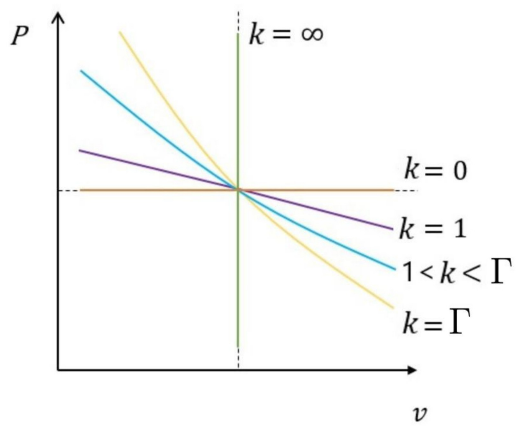

Figure 1.6: Polytropic processes

This finesses the issue of the generation and flow of heat in stellar interiors, which determines the temperature $T(r)$ and hence the pressure $P(\rho_0, T)$.

$$\begin{aligned}
\text{Low-mass white-dwarf stars, } n &= 1.5 \text{ polytropes } (\Gamma = 1.666) \\
\text{red-giant stars, } n &= 3 \quad \text{polytropes } (\Gamma = 1.333) \\
\text{giant planets (Jupiter, Saturn), } n &= 1 \quad \text{polytropes } (\Gamma = 2) \\
\text{small planets (Mercury), } n &\approx 0 \quad \text{polytropes } (\Gamma = \infty)
\end{aligned} \tag{1.19}$$

An idealization that is often accurate in fluid dynamics is that the fluid is adiabatic: no heating or cooling from dissipative processes, such as viscosity, thermal conductivity, or the emission and absorption of radiation. When this is a good approximation, the entropy per unit mass s of a fluid element is constant.

1.3 Hydrostatics

The equation of hydrostatic equilibrium for a fluid at rest in a gravitational field **g** is the same as the equation of elastostatic equilibrium with a vanishing shear stress,

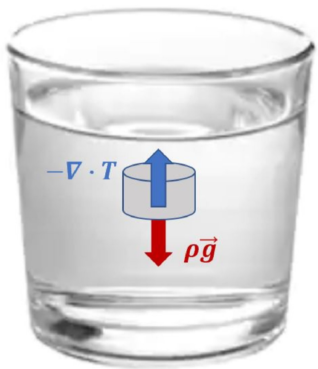

Figure 1.7: Hydrostatic equilibrium

(1) hydrostatic equilibrium:

$$\boxed{\mathbf{f}_{\text{on fluid}} + \mathbf{f}_{\text{external}} = 0 = \mathbf{f}_{\text{net}}} \;\Rightarrow\; \boxed{-\boldsymbol{\nabla}\cdot\mathsf{T}_{\text{fluid}} = -\rho_0\vec{g}}$$

$$\xrightarrow{\mathsf{T}_{\text{fluid}} = P\mathbf{g}} \boxed{\boldsymbol{\nabla}P = \rho_0\vec{g}}$$

$$\xrightarrow{\vec{g} = -\boldsymbol{\nabla}\Phi} \boxed{\boldsymbol{\nabla}P = \rho_0\vec{g} = -\rho_0\boldsymbol{\nabla}\Phi}$$

(2) constant density (isochores) on the equipotential surface:

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}P = -\rho_0\boldsymbol{\nabla}\Phi)$$

$$\hookrightarrow \boldsymbol{\nabla}\rho_0\times\boldsymbol{\nabla}\Phi = 0 \;\Rightarrow\; \boldsymbol{\nabla}\rho_0 \parallel \boldsymbol{\nabla}\Phi$$

$$\hookrightarrow \rho_0 = \rho_0(\Phi)$$

(3) constant pressure (the isobars) on the equipotential surface:

$$\boldsymbol{\nabla}P = -\rho_0\boldsymbol{\nabla}\Phi \;\rightarrow\; \boldsymbol{\nabla}P\propto\boldsymbol{\nabla}\Phi$$

$$\hookrightarrow \mathrm{d}P = -\rho_0\mathrm{d}\Phi$$

$$\hookrightarrow \Delta P = -\int_{\Phi_1}^{\Phi_2}\rho_0(\Phi)\mathrm{d}\Phi \tag{1.20}$$

For example, the deepest point in the world's oceans is the bottom of the Mariana Trench in the Pacific, 11 km below sea level. Calculating the pressure at the bottom,

$$P(z) = g\int_z^{\infty}\rho_0\mathrm{d}z = (10\mathrm{m/s^2})\int_{-11\mathrm{km}}^{0\mathrm{km}}(10^3\mathrm{kg/m^3})\mathrm{d}z = 10^8\mathrm{Pa} = 10^3\mathrm{atm}\,. \tag{1.21}$$

The equation of hydrostatic equilibrium can be applied in various situations.

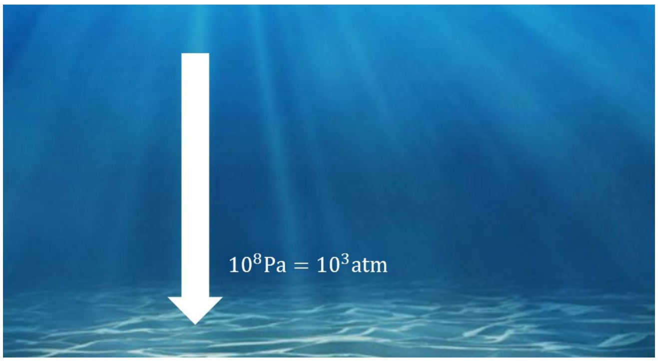

Figure 1.8: The Mariana Trench in the Pacific

1.3.1 Archimedes' Law (Buoyant force)

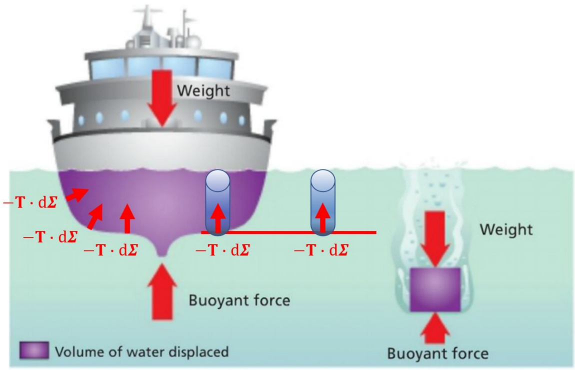

Figure 1.9: Archimedes' law

Archimedes' law states that, whether a solid body is fully or partially submerged in a uniform gravitational field $\vec{g} = -g\hat{z}$, the upward buoyant force of the fluid on the body is equal to the weight of the displaced fluid. Since the equipotential surface is parallel to the constant pressure surface, we can replace the solid body by fluid without changing the pressure on the surface $\partial\mathcal{V}$. In this way, we can derive the Archimedes' law as follows.

(1) hydrostatic equilibrium:

$$\boxed{\mathbf{f}_{\text{on fluid}} + \mathbf{f}_{\text{external}} = 0 = \mathbf{f}_{\text{net}}} \;\Rightarrow\; \boxed{-\boldsymbol{\nabla}\cdot\mathbf{T}_{\text{fluid}} = -\rho_0\vec{g}}$$

$$\xrightarrow{\mathbf{T}_{\text{fluid}} = P\mathbf{g}} \boxed{\boldsymbol{\nabla}P = \rho_0\vec{g}}$$

$$\xrightarrow{\vec{g} = -\boldsymbol{\nabla}\Phi} \boxed{\boldsymbol{\nabla}P = \rho_0\vec{g} = -\rho_0\boldsymbol{\nabla}\Phi}$$

(2)
$$\begin{aligned}
\mathbf{F}^{\text{buoy}} &= -\int_{\partial\mathcal{V}} \mathbf{T}\cdot d\boldsymbol{\Sigma} \\
&= -\int_{\mathcal{V}} \boldsymbol{\nabla}\cdot\mathbf{T}\,\mathrm{d}V \\
&\quad \boldsymbol{\nabla}\cdot\mathbf{T}_{\text{fluid}} = \rho_0\vec{g}\ \text{(hydrostatic equilibrium)} \\
&= -\vec{g}\int_{\mathcal{V}} \rho_0\, dV \\
&= -M\vec{g}
\end{aligned} \tag{1.22}$$

1.3.2 Non-rotating stars and planets

For non-rotating stars, we can consider a spherical, self-gravitating fluid body in spherical polar coordinates:

(1) hydrostatic equilibrium:

$$\boxed{\mathbf{f}_{\text{on fluid}} + \mathbf{f}_{\text{external}} = 0 = \mathbf{f}_{\text{net}}} \Rightarrow \boxed{-\boldsymbol{\nabla}\cdot\mathbf{T}_{\text{fluid}} + \rho_0\vec{g} = 0}$$

$$\xrightarrow{\mathbf{T}_{\text{fluid}} = P\mathbf{g}} \boxed{\boldsymbol{\nabla}P = \rho_0\vec{g}}$$

$$\xrightarrow{\vec{g} = -\boldsymbol{\nabla}\Phi} \boxed{\boldsymbol{\nabla}P = \rho_0\vec{g} = -\rho_0\boldsymbol{\nabla}\Phi}$$

(2) $\boldsymbol{\nabla}P = -\rho_0\boldsymbol{\nabla}\Phi$

$$P = P(r),\ \Phi = \Phi(r)$$

$$\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\varphi}\hat{\varphi}$$

$$\hookrightarrow \frac{\mathrm{d}P}{\mathrm{d}r} = -\rho_0\frac{\mathrm{d}\Phi}{\mathrm{d}r} \tag{1.23}$$

Then, from Poisson's equation, we get an alternative form of the equation of hydrostatic equilibrium for a non-rotating star as follows.

$$\nabla^2\Phi = 4\pi G\rho_0$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}$$

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = 4\pi G\rho_0$$

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho_0\frac{\mathrm{d}\Phi}{\mathrm{d}r} \text{ (hydrostatic equilibrium for spherical objects)}$$

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho_0}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = -4\pi G\rho_0$$

$$\times r^2\mathrm{d}r,\ \int$$

$$\int \mathrm{d}\left(\frac{r^2}{\rho_0}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = \left[\frac{r^2}{\rho_0}\frac{\mathrm{d}P}{\mathrm{d}r}\right]_0^r = -\int_0^r 4\pi G\rho_0 r^2\mathrm{d}r$$

$$\left.\frac{\mathrm{d}P}{\mathrm{d}r}\right|_{r=0} = 0 \text{ to avoid a singular pressure at center}$$

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{\rho_0}{r^2}G\underbrace{\int_0^r \rho_0(4\pi r^2)\mathrm{d}r}_{\equiv m(r)}$$

$$\hookrightarrow \boxed{\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho_0 \frac{Gm(r)}{r^2}} \tag{1.24}$$

This is a good approximation for solid planets (e.g., Earth) as well as for stars and fluid planets (e.g., Jupiter) because, at the enormous stresses encountered in the interior of a solid planet, the strains are so large that plastic flow occurs. In other words, the shear stresses are much smaller than the isotropic part of the stress tensor.

1.3.3 Lane-Emden equation of stellar structure

The polytropes which indicate nonrotating, spherical fluid bodies with the polytropic equation of state $P = K\rho_0^{1+1/n}$ gives a useful equation describing the stellar structure, which is called the Lane-Emden equation:

$$(1)\ \nabla^2\Phi = 4\pi G\rho_0 \ \rightarrow\ \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = 4\pi G\rho_0$$

$$\hookrightarrow \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho_0}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = -4\pi G\rho_0 \ \cdots (A)$$

$$\hookrightarrow \frac{\mathrm{d}P}{\mathrm{d}r} = -\rho_0\frac{Gm(r)}{r^2}, \quad \left(m(r) = \int_0^r 4\pi\rho_0 r^2 \mathrm{d}r\right)$$

$$(2)\ (A) \rightarrow \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho_0}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = -4\pi G\rho_0$$

$$P = K\rho_0^{1+1/n} \xrightarrow{\rho_0 \equiv \rho_{0c}\theta^n} K\rho_c^{1+1/n}\theta^{1+n}$$

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\underbrace{\frac{r^2}{\rho}}_{=\frac{r^2}{\rho_{0c}\theta^n}}\underbrace{\frac{\mathrm{d}P}{\mathrm{d}r}}_{=K\rho_{0c}^{1+1/n}(1+n)\theta^n\frac{\mathrm{d}\theta}{\mathrm{d}r}}\right) = -4\pi G\overbrace{\rho_0}^{=\rho_{0c}\theta^n}$$

$$\hookrightarrow \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 K\rho_{0c}^{1/n}(1+n)\frac{\mathrm{d}\theta}{\mathrm{d}r}\right) = -4\pi G\rho_{0c}\theta^n$$

$$r \equiv a\xi, \quad \times\frac{1}{4\pi G\rho_{0c}}$$

$$\hookrightarrow \frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{1}{a^2}\frac{K\rho_{0c}^{1/n}(1+n)}{4\pi G\rho_{0c}}\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n$$

$$a^2 \equiv \frac{K\rho_{0c}^{1/n}(1+n)}{4\pi G\rho_{0c}}$$

$$\hookrightarrow \frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n \tag{1.25}$$

Therefore, the Lane-Emden equation is given by

Lane-Emden equation:

$$\boxed{\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n}$$

$$\text{where } \xi = a^{-1}r,\ a^2 = \frac{K\rho_{0c}^{1/n}(1+n)}{4\pi G\rho_{0c}} \tag{1.26}$$

The boundary condition $\frac{\mathrm{d}P}{\mathrm{d}r}\big|_{r=0} = 0$ becomes

boundary condition of Lane-Emden equation:

$$\begin{cases} \rho_0(\xi = r = 0) \equiv \rho_{0c} \xrightarrow{\rho_0 \equiv \rho_{0c}\theta^n} \boxed{\theta(\xi = r = 0) = 1 \text{ at the center}} \\ \\ \left.\frac{\mathrm{d}P}{\mathrm{d}r}\right|_{r=0} = 0 \xrightarrow[\rho_0 \equiv \rho_{0c}\theta^n,\ \xi = a^{-1}r]{P = K\rho_0^{1+1/n}} \boxed{\theta'(\xi = r = 0) = 1 \text{ at the center}} \end{cases} \tag{1.27}$$

By integrating the Lane-Emden equation, from $\xi = r = 0$ until $\xi_1(R)$ where $\rho_0 = P = 0$, we can find the polytrope's surface with a physical radius $R = a\xi_1$ and its mass $M = \int_0^R \rho_0(4\pi r^2)\mathrm{d}r$.

$$R = \left[\frac{(n+1)K}{4\pi G}\right]^{1/2} \rho_{0c}^{(1-n)/2n}\xi_1$$

$$M = \int_0^R \underbrace{\rho_0}_{=\rho_{0c}\theta^n = -\rho_{0c}\frac{1}{\cancel{\xi^2}}\frac{\mathrm{d}}{\mathrm{d}\xi}\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi} \text{ (by L-E eq.)}} (4\pi \overbrace{r^2}^{=\alpha^2\cancel{\xi^2}}) \overbrace{\mathrm{d}r}^{=\mathrm{d}(\alpha\xi)} = 4\pi R^2\alpha^3(R)\rho_{0c}(-\xi^2\theta')|_R$$

in terms of R by using $R = \left[\frac{(n+1)K}{4\pi G}\right]^{1/2} \rho_{0c}^{(1-n)/2n}\xi_1$

$$= 4\pi R^{(3-n)/(1-n)}\left[\frac{(n+1)K}{4\pi G}\right]^{n/(n-1)} \xi_1^{(n+1)/(n-1)}|\theta'(\xi_1)| \tag{1.28}$$

Analytic solutions for this Lane-Emden equation are known for $n = 0, 1, 5$.

$$\begin{cases} n = 0 & \theta = 1 - \frac{\xi 2}{6}, \qquad \xi_1 = \sqrt{6} \\ n = 1 & \theta = \frac{\sin\xi}{\xi}, \qquad \xi_1 = \pi \\ n = 5 & \theta = \left(1 + \frac{\xi^2}{3}\right)^{-\frac{1}{2}}, \quad \xi_1 = \infty \end{cases} \tag{1.29}$$

For example, Jupiter and Saturn are both made up of an H-He fluid that is well approximated by a polytrope of index $n = 1$, $P = K\rho_0^2$, with the same constant K. Using the information that $M_{\text{J}} = 2 \times 10^{27}$kg, $R_{\text{J}} = 7 \times 10^4$km, and $M_{\text{S}} = 6 \times 10^{26}$kg, we can estimate the radius of Saturn. Since the Lane-Emden equation has a simple analytic solution, $\theta = \sin\xi/\xi$, we can also compute the central density of Jupiter and Saturn.

1.3.4 Rotating star

The equation of hydrostatic equilibrium is readily extended to bodies rotating rigidly with a uniform angular velocity   relative to an inertial frame. In a corotating frame with the body, the fluid velocity vanishes and the equation of hydrostatic equilibrium is changed by the centrifugal force per unit volume as:

(1) hydrostatic equilibrium:

$$\boxed{\mathbf{f}_{\text{on fluid}} + \mathbf{f}_{\text{external}} = 0 = \mathbf{f}_{\text{net}}} \;\Rightarrow\; \boxed{-\boldsymbol{\nabla}\cdot\mathbf{T}_{\text{fluid}} = -\rho_0\vec{g}}$$

$$\xrightarrow{\mathbf{T}_{\text{fluid}} = P\mathbf{g}} \boxed{\boldsymbol{\nabla}P = \rho_0\vec{g}}$$

$$\xrightarrow{\vec{g} = -\boldsymbol{\nabla}\Phi} \boxed{\boldsymbol{\nabla}P = \rho_0\vec{g} = -\rho_0\boldsymbol{\nabla}\Phi}$$

(2) in the corotating frame:

$$\boxed{\boldsymbol{\nabla}P = \rho_0(\vec{g} + \vec{g}_{\text{cen}}) = -\rho_0\boldsymbol{\nabla}(\Phi + \Phi_{\text{cen}})}$$

$$\text{where } \vec{g}_{\text{cen}} = - \;\times (\;\times \mathbf{r}) = -\boldsymbol{\nabla}\Phi_{\text{cen}} \;\rightarrow\; \Phi_{\text{cen}} = -\frac{1}{2}(\;\times \mathbf{r})^2 \tag{1.30}$$

By using the above relation, we can calculate the discrepancy between the equatorial and polar radii of a planet. For example,

(1) The surface of a planet is a equipotential surface of $\Phi + \Phi_{\text{cen}}$.

(2) $R_e - R_p \simeq \dfrac{\Omega^2 R^2}{2g}$

(3) for Earth, $g \simeq 10\,\mathrm{m\,s^{-2}},\ R \simeq 6 \times 10^6\mathrm{m},\ \Omega \simeq 7 \times 10^{-5}\mathrm{rad\,s^{-1}}$

$$\Delta R_{\mathrm{Earth}} \sim 10\,\mathrm{km} \text{ (The correct value is } 21\mathrm{km.)} \tag{1.31}$$

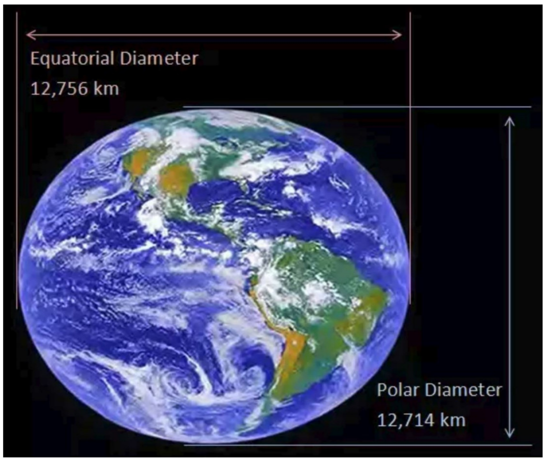

Figure 1.10: An Oblate shape of the Earth

1.3.5 Conservation laws (Mass)

The total charge and the total particle number inside a 3-dimensional region $\mathcal{V}$ are

$$\begin{cases} \text{(total charge)} & Q = \int_{\mathcal{V}} \rho_{\mathrm{e}} dV \\ \text{(total number)} & N = \int_{\mathcal{V}} n dV \end{cases} . \tag{1.32}$$

When we denote the rate of the charge and particles flowing out through the boundary of $\mathcal{V}$, $\partial\mathcal{V}$ as **j** and **S**, the integral laws of charge conservation is written by

$$\begin{cases} \text{(charge conservation)} & \dfrac{\mathrm{d}Q_{\text{in}}}{\mathrm{d}t} + \dfrac{\mathrm{d}Q_{\text{out}}}{\mathrm{d}t} = \dfrac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}} \rho_{\text{e}} dV + \underbrace{\int_{\partial\mathcal{V}} \mathbf{j}\cdot d\mathbf{\Sigma}}_{=\int_{\mathcal{V}} \nabla\cdot\mathbf{j}dV} = 0 \\ \text{(number conservation)} & \dfrac{\mathrm{d}N_{\text{in}}}{\mathrm{d}t} + \dfrac{\mathrm{d}N_{\text{out}}}{\mathrm{d}t} = \dfrac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}} n dV + \underbrace{\int_{\partial\mathcal{V}} \mathbf{S}\cdot d\mathbf{\Sigma}}_{=\int_{\mathcal{V}} \nabla\cdot\mathbf{S}dV} = 0 \end{cases} \tag{1.33}$$

Then, by considering the volume element time-independent and by using Stokes' theorem, we get the differential form of the conservation equation:

$$\begin{cases} \text{fixed volume element} & : \dfrac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}} \rho_{\text{e}} dV \to \int_{\mathcal{V}} \dfrac{\partial\rho_{\text{e}}}{\partial t} dV \\ \text{Stokes' theorem} & : \int_{\partial\mathcal{V}} \mathbf{j}\cdot d\mathbf{\Sigma} = \int_{\mathcal{V}} \nabla\cdot\mathbf{j}dV \end{cases}$$

$$\hookrightarrow \begin{cases} \text{(charge conservation)} & \dfrac{\partial\rho_{\text{e}}}{\partial t} + \nabla\cdot\mathbf{j} = 0 \\ \text{(number conservation)} & \dfrac{\partial n}{\partial t} + \nabla\cdot\mathbf{S} = 0 \end{cases}$$

$\Rightarrow$ (conservation law) :

$$\boxed{\frac{\partial}{\partial t}\text{(some density)} + \nabla\cdot\text{(density flux)} = 0} \tag{1.34}$$

In the differential form of the conservation law, we fixed the position of volume elements and the partial derivative with respect to time indicates the density change at the fixed location.

Euler approach:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}} \rho_0 \mathrm{d}V = -\int_{\partial\mathcal{V}} \rho_0 \vec{v}\cdot d\vec{\Sigma}}$$

$dV \neq dV(t)$, by using Stokes' theorem

$$\int_{\mathcal{V}} \underbrace{\frac{\partial\rho_0}{\partial t}}_{\text{:Eulerian derivative}} \mathrm{d}V = -\int_{\mathcal{V}} \vec{\nabla}\cdot(\rho_0\vec{v})\mathrm{d}V$$

$$\hookrightarrow \boxed{\frac{\partial\rho_0}{\partial t} + \nabla\cdot(\rho_0\mathbf{v}) = 0} \tag{1.35}$$

This time derivative and approach are called the Eulerian derivative and the Eulerian approach. There is an alternative approach. When we assume the volume element is changing along the flow, we can write the conservation equation in another form.

Lagrangian approach:

$$\boxed{\frac{\mathrm{d}M}{\mathrm{d}t} = 0 = \frac{\mathrm{d}}{\mathrm{d}t}\int \rho_0 \delta V(t)}$$
$$= \int \frac{\mathrm{d}\rho_0}{\mathrm{d}t}\delta x\delta y\delta z + \int \rho_0 \underbrace{(\underbrace{\frac{\mathrm{d}\delta x}{\mathrm{d}t}}_{=(\delta v_x/\delta x)\delta x} \delta y\delta z + \cdots)}_{=(\boldsymbol{\nabla}\cdot\mathbf{v})\delta V}$$
$$= \int \left[\frac{\mathrm{d}\rho_0}{\mathrm{d}t} + \rho_0(\boldsymbol{\nabla}\cdot\mathbf{v})\right]\delta V = 0$$
$$\hookrightarrow \boxed{\frac{\mathrm{d}\rho_0}{\mathrm{d}t} + \rho_0\boldsymbol{\nabla}\cdot\mathbf{v} = 0}$$

convective time derivative:

$$\text{since } \frac{\partial\rho_0}{\partial t} = -\boldsymbol{\nabla}\cdot(\rho_0\mathbf{v}) = -\mathbf{v}\cdot\boldsymbol{\nabla}\rho_0 - \rho_0\boldsymbol{\nabla}\cdot\mathbf{v}$$
$$\hookrightarrow \frac{\mathrm{d}\rho_0}{\mathrm{d}t} = -\rho_0\boldsymbol{\nabla}\cdot\mathbf{v} = \frac{\partial\rho_0}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\rho_0$$
$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}} \quad \text{: convective(advective) time derivative} \tag{1.36}$$

In the above explanation, the equation of the Lagrangian approach can be derived simply from the Eulerian relation by differentiating $\rho_0\mathbf{v}$ as

$$\text{mass conservation: } \frac{\partial\rho_0}{\partial t} + \boldsymbol{\nabla}\cdot(\rho_0\mathbf{v}) = 0 \text{ (Eulerian approach)}$$
$$\hookrightarrow \underbrace{\frac{\partial\rho_0}{\partial t} + (\boldsymbol{\nabla}\rho_0)\cdot\mathbf{v}}_{=\frac{\mathrm{d}\rho_0}{\mathrm{d}t}} + \rho_0(\boldsymbol{\nabla}\cdot\mathbf{v}) = 0$$
$$\hookrightarrow \frac{\mathrm{d}\rho_0}{\mathrm{d}t} + \rho_0(\boldsymbol{\nabla}\cdot\mathbf{v}) = 0 \text{ (Lagrangian approach)} \tag{1.37}$$

As you can see in the derivation, in the Lagrangian approach, we take the time derivative of the density of the volume element moving with the fluid.

Considering a small fluid element, we understand the divergent of the velocity vector is the fluid's rate of expansion.

$$
\begin{aligned}
&\delta M = \rho_0 \delta V \\
&\xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}} \frac{\mathrm{d}\delta M}{\mathrm{d}t} = \frac{\mathrm{d}\rho_0}{\mathrm{d}t}\delta V + \rho_0 \frac{\mathrm{d}\delta V}{\mathrm{d}t} = 0 \\
&\quad \hookrightarrow \frac{\mathrm{d}\rho_0}{\mathrm{d}t} = -\rho_0 \frac{\mathrm{d}(\delta V)/\mathrm{d}t}{V} \Leftrightarrow -\rho_0 \boldsymbol{\nabla} \cdot \mathbf{v} \\
&\therefore \boxed{\boldsymbol{\nabla} \cdot \mathbf{v} = \frac{\mathrm{d}(\delta V)/\mathrm{d}t}{V} = \frac{\mathrm{d}\Theta}{\mathrm{d}t}}
\end{aligned} \tag{1.38}
$$

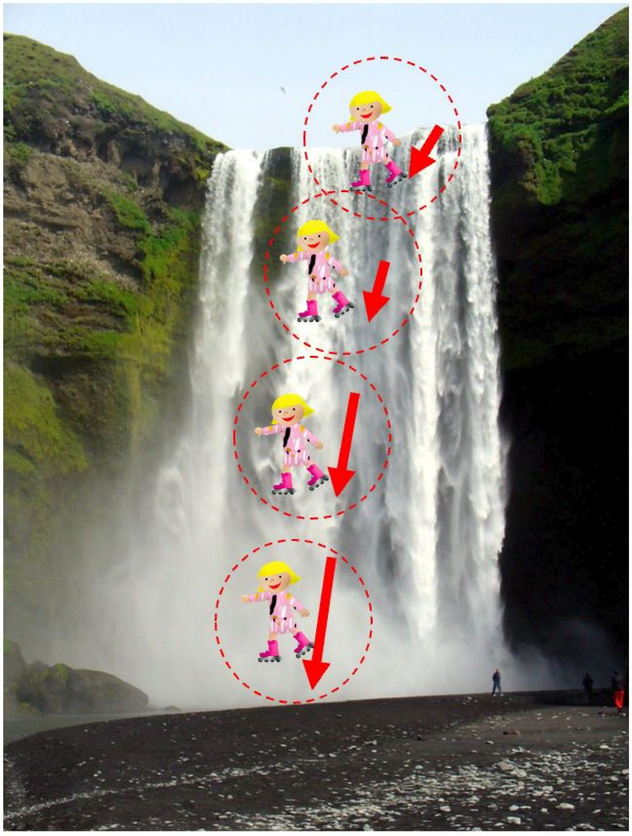

Figure 1.11: Eulerian and Lagrangian approach

1.3.6 Conservation law (Momentum)

When there is no external force, the mechanical momentum flux is, by definition, the stress tensor T, and the differential conservation law written by

Eulerian approach:

when there's no external force on the fluid element,

$$\begin{cases} \mathbf{F}_\mathrm{m} = \dfrac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int_{\mathcal{V}} (\rho_0 \mathbf{v}) dV = \int_{\mathcal{V}} \dfrac{\partial(\rho_0 \mathbf{v})}{\partial t} dV \\ \mathbf{F}_\mathrm{m} = \displaystyle\int \mathbf{f}_\mathrm{m} dV = \int \boldsymbol{\nabla} \cdot \mathbf{T} dV \end{cases}$$

$$\Rightarrow \boxed{\frac{\partial(\rho_0 \mathbf{v})}{\partial t} + \boldsymbol{\nabla} \cdot [(\rho_0 \mathbf{v}) \otimes \mathbf{v}] = \frac{\partial(\rho_0 \mathbf{v})}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{T}_\mathrm{m} = 0} \tag{1.39}$$

Then, we find the mechanical stress tensor of the fluid as

$$\mathbf{T}_\mathrm{m} = \rho_0 \mathbf{v} \otimes \mathbf{v} \tag{1.40}$$

In the presence of external forces, the net force acts on a fluid. By denoting the net force density as $\mathbf{f}_\mathrm{net}$, we obtain the Lagrangian approach relation, which is just Newton's law per unit volume.

$$\begin{aligned} \mathbf{f}_\mathrm{net} &= \underbrace{\frac{\partial(\rho_0 \mathbf{v})}{\partial t}}_{= \frac{\partial \rho_0}{\partial t}\mathbf{v} + \rho_0 \frac{\partial \mathbf{v}}{\partial t}} + \underbrace{\boldsymbol{\nabla} \cdot \mathbf{T}_\mathrm{m}}_{= \boldsymbol{\nabla} \cdot (\rho_0 \mathbf{v} \otimes \mathbf{v}) = \boldsymbol{\nabla} \cdot (\rho_0 \mathbf{v})\mathbf{v} + (\rho_0 \mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v}} \\ &= \cancel{\frac{\partial \rho_0}{\partial t}}\mathbf{v} + \rho_0 \frac{\partial \mathbf{v}}{\partial t} + \cancel{\boldsymbol{\nabla} \cdot (\rho_0 \mathbf{v})}\mathbf{v} + (\rho_0 \mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} \\ &\quad \text{by mass conservation, } \frac{\partial \rho_0}{\partial t} + \boldsymbol{\nabla} \cdot (\rho_0 \mathbf{v}) = 0 \\ &= \rho_0 \frac{\partial \mathbf{v}}{\partial t} + (\rho_0 \mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} \\ &= \rho_0 \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \end{aligned} \tag{1.41}$$

By writing down the stress tensor of the external force density, we find the Eulerian momentum conservation equation:

$$\begin{aligned} &\frac{\partial(\rho_0 \mathbf{v})}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{T}_\mathrm{m} = \mathbf{f}_\mathrm{net} \equiv -\boldsymbol{\nabla} \cdot \mathbf{T}_\mathrm{f} \\ &\quad \mathbf{T} \equiv \mathbf{T}_\mathrm{m} + \mathbf{T}_\mathrm{f} \end{aligned}$$

$$\Rightarrow \boxed{\frac{\partial(\rho_0 \mathbf{v})}{\partial t} + \nabla \cdot \mathbf{T} = 0} \tag{1.42}$$

1.3.7 Conservation law (Energy)

For a fluid with an energy density $U(\mathbf{x}, t)$ and an energy flux $\mathbf{F}(\mathbf{x}, t)$, we proceed in the same way as we did for mass and momentum. The energy conservation law is written by

$$\frac{\partial U}{\partial t} + \nabla \cdot \mathbf{F} = 0 \tag{1.43}$$

1.4 Hydrodynamics

1.4.1 Fluid stress tensor in hydrodynamics

In this section, we deal with an ideal fluid flowing without dissipative processes (viscosity and thermal conductivity), that is, without the entropy change of a fluid element. When the fluid flows by the external force, the mass and momentum conservation laws are the same as we explained in the previous section.

When there is an external force, and the fluid flows,

$$\begin{aligned} &\text{mass conservation: } \frac{\partial \rho_0}{\partial t} + \nabla \cdot (\rho_0 \mathbf{v}) = 0 \\ &\text{momentum conservation: } \frac{\partial(\rho_0 \mathbf{v})}{\partial t} + \nabla \cdot \underbrace{\mathbf{T}}_{\text{but this will be changed.}} = 0 \end{aligned} \tag{1.44}$$

When fluid flows slowly compared to its sound speed, and a gravitational effect is modest, the fluid density remains nearly constant. In this case, we can use the incompressible approximation, $\nabla \cdot \mathbf{v} = 0$.

For an ideal fluid, the only forces acting on the fluid are a gravitational force and the fluid's pressure. Then the momentum conservation law is written by

(1) **hydrostatic equilibrium:**

$$\boxed{\mathbf{f}_{\text{on fluid}} + \mathbf{f}_{\text{external}} = 0 = \mathbf{f}_{\text{net}}} \;\Rightarrow\; \boxed{-\boldsymbol{\nabla}\cdot\mathbf{T}_{\text{fluid}} + \rho_0\vec{g} = 0}$$

$$\xrightarrow{\mathbf{T}_{\text{fluid}} = P\mathbf{g}} \boxed{\boldsymbol{\nabla}P = \rho_0\vec{g}}$$

$$\xrightarrow{\vec{g} = -\boldsymbol{\nabla}\Phi} \boxed{\boldsymbol{\nabla}P = \rho_0\vec{g} = -\rho_0\boldsymbol{\nabla}\Phi}$$

(2) when there is no net force on a fluid element,

$$\frac{\partial(\rho_0\mathbf{v})}{\partial t} + \boldsymbol{\nabla}\cdot[(\rho_0\mathbf{v})\otimes\mathbf{v}] = \frac{\partial(\rho_0\mathbf{v})}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{T}_{\text{m}} = \rho_0\frac{\text{d}\mathbf{v}}{\text{d}t} = \mathbf{f}_{\text{net}} = 0$$

(3) when there is a net force on a fluid element,

hydrodynamics: $\mathbf{f}_{\text{on fluid}} + \mathbf{f}_{\text{external}} = \mathbf{f}_{\text{net}} \neq 0$

$$\frac{\partial(\rho_0\mathbf{v})}{\partial t} + \boldsymbol{\nabla}\cdot[(\rho_0\mathbf{v})\otimes\mathbf{v}] = \mathbf{f}_{\text{net}} = \underbrace{-\nabla P}_{\text{for ideal fluid}} + \rho_0\vec{g}$$

$$\Rightarrow \frac{\partial(\rho_0\mathbf{v})}{\partial t} + \boldsymbol{\nabla}\cdot\underbrace{(\rho_0\mathbf{v}\otimes\mathbf{v} + P\mathbf{g})}_{\mathbf{T}_{\text{ideal fluid}}} = \rho_0\vec{g}$$

$$\therefore \boxed{\frac{\partial(\rho_0\mathbf{v})}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{T}_{\text{ideal fluid}} = \rho_0\vec{g}}$$

(3) stress tensor in hydrodynamics:

$$\boxed{\mathbf{T}_{\text{fluid}}:\; P\mathbf{g} \;\rightarrow\; \rho_0\mathbf{v}\otimes\mathbf{v} + P\mathbf{g}} \tag{1.45}$$

1.4.2 Euler equation

The Euler equation, named after Leonhard Euler, is hyperbolic equations governing adiabatic and inviscid (zero viscosity) flow. By using the obtained equations in the previous sections, which are the mass and momentum conservation laws of fluids, we find the following equation:

$$\underbrace{\frac{\partial(\rho_0\mathbf{v})}{\partial t}}_{=\frac{\partial\rho_0}{\partial t}\mathbf{v} + \rho_0\frac{\partial\mathbf{v}}{\partial t}} + \underbrace{\boldsymbol{\nabla}\cdot(\rho_0\mathbf{v}\otimes\mathbf{v} + P\mathbf{g})}_{=[\boldsymbol{\nabla}\cdot(\rho_0\mathbf{v})]\mathbf{v} + \rho_0(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}} = \rho_0\vec{g} \quad \text{(momentum conservation)}$$

$$\frac{\partial\rho_0}{\partial t} + \boldsymbol{\nabla}\cdot(\rho_0\mathbf{v}) = 0 \text{ (by using the mass conservation equation)}$$

$$\hookrightarrow \cancel{\frac{\partial \rho_0}{\partial t}}\mathbf{v} + \rho_0 \frac{\partial \mathbf{v}}{\partial t} + [\cancel{\boldsymbol{\nabla} \cdot (\rho_0 \mathbf{v})}]\mathbf{v} + \rho_0(\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} + \boldsymbol{\nabla} P = \rho_0 \vec{g}$$

$$\times \frac{1}{\rho_0}$$

$$\hookrightarrow \boxed{\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \equiv \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\frac{\boldsymbol{\nabla} P}{\rho_0} + \vec{g} \text{ for an ideal fluid}} \tag{1.46}$$

This equation is called the Euler equation. The convective derivative of the velocity is taken moving its location along with the fluid flow. Thus, it is the acceleration felt by an observer who is flowing with the fluid.

Now we have four differential equations (the three Euler equation and one mass conservation equation) and five unknowns (ρ_0, P, $\mathbf{v}$). Another equation we need is the equation which specifies a relation between the thermodynamic properties, such as ρ_0, P, and s. For an ideal fluid, we have P as a function of ρ_0, which is the result of the conserved entropy of each fluid element. In practice, the equation of state is often well approximated by incompressibility, ρ_0=constant, or by a polytropic relation, $P = K(s)\rho_0^{1+1/n}$.

(1) five unknowns: ($\rho_0,\ P,\ \mathbf{v}$)

(2) four differential equations

$$\begin{cases} \text{mass conservation} & \frac{\partial \rho_0}{\partial t} + \boldsymbol{\nabla} \cdot (\rho_0 \mathbf{v}) = 0 \ \rightarrow \text{ one DE} \\ \text{Euler equation} & \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\frac{\boldsymbol{\nabla} P}{\rho_0} + \vec{g} \quad \rightarrow \text{three DE} \end{cases}$$

(3) one more relation: equation of state

$$\begin{cases} \text{for an ideal fluid} & P(\rho_0) \\ \text{incompressible approximation} & \rho_0 = \text{constant} \\ \text{polytropes} & P = K(s)\rho_0^{1+1/n} \\ \quad \vdots & \quad \vdots \end{cases}$$

1.4.3 Bernoulli's theorem

Introducing the vorticity which is twice the angular velocity of rotation of a fluid element,

$$(1)\text{ Curl is two times of angular velocity:}$$

$$\boldsymbol{\omega} \equiv \boldsymbol{\nabla} \times \mathbf{v} = 2\frac{\mathrm{d}\phi}{\mathrm{d}t}$$

$$\text{(ex) for } \hat{\mathbf{z}} \text{ rotation, } (\boldsymbol{\nabla} \times \mathbf{v}) = \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}} = 2\omega\hat{\mathbf{z}}$$

$$(2)\ \mathbf{v} \times \boldsymbol{\omega} \equiv \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{v}) = \frac{1}{2}\boldsymbol{\nabla}v^2 - (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v}$$

$$\boldsymbol{\nabla}(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{B}) + (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B} + (A \leftrightarrow B)$$

$$(\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} = \mathbf{v} \times \boldsymbol{\omega} - \frac{1}{2}\boldsymbol{\nabla}v^2 \tag{1.47}$$

we rewrite the Euler equation in terms of the angular velocity:

$$(1)\text{ Euler equation:}$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} = -\frac{\boldsymbol{\nabla}P}{\rho_0} + \vec{g}$$

$$(\mathbf{v} \cdot \nabla)\mathbf{v} = -\mathbf{v} \times \omega + \frac{1}{2}\nabla v^2, \quad \vec{g} = -\nabla\Phi$$

$$\frac{\partial \mathbf{v}}{\partial t} - \mathbf{v} \times \boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\nabla}v^2 = -\frac{\boldsymbol{\nabla}P}{\rho_0} - \boldsymbol{\nabla}\Phi$$

$$\therefore \boxed{\frac{\partial \mathbf{v}}{\partial t} - \boldsymbol{\nabla}\left(\frac{1}{2}v^2 + \Phi\right) + \frac{\boldsymbol{\nabla}P}{\rho_0} - \mathbf{v} \times \boldsymbol{\omega} = 0} \tag{1.48}$$

This is the most general version of Bernoulli's theorem—valid for any ideal fluid. For steady flow ($(\partial/\partial t)$(any quantity) $= 0$) of an ideal fluid which satisfies $\mathrm{d}s/\mathrm{d}t = \cancel{\partial s/\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})s = 0$, the thermodynamic identity holds:

$$\mathrm{d}h = T\mathrm{d}s + \rho_0^{-1}\mathrm{d}P$$

$$\mathrm{d}X \to (\mathbf{v} \cdot \boldsymbol{\nabla})X$$

$$\hookrightarrow (\mathbf{v} \cdot \boldsymbol{\nabla})h = \underbrace{\cancel{T(\mathbf{v} \cdot \boldsymbol{\nabla})s}}_{\because\ \mathrm{d}s/\mathrm{d}t = \cancel{\partial s/\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})s = 0} + \rho_0^{-1}(\mathbf{v} \cdot \boldsymbol{\nabla})P \ \cdots (1) \tag{1.49}$$

Then from the equation of Bernoulli's theorem, we get

$$\frac{\partial \mathbf{v}}{\partial t} - \boldsymbol{\nabla}\left(\frac{1}{2}v^2 + \Phi\right) + \frac{\boldsymbol{\nabla}P}{\rho_0} - \mathbf{v} \times \boldsymbol{\omega} = 0$$

$\curvearrowright$ $\mathbf{v}\cdot$

$$\mathbf{v}\cdot\left[\frac{\partial\mathbf{v}}{\partial t}-\boldsymbol{\nabla}\left(\frac{1}{2}v^2+\Phi\right)+\frac{\boldsymbol{\nabla}P}{\rho_0}-\mathbf{v}\times\boldsymbol{\omega}=0\right]$$

$$\hookrightarrow \underbrace{\mathbf{v}\cdot\frac{\partial\cancel{\mathbf{v}}}{\cancel{\partial t}}}_{\text{for steady flow}}-\mathbf{v}\cdot\boldsymbol{\nabla}\left(\frac{1}{2}v^2+\Phi\right)+\underbrace{\frac{(\mathbf{v}\cdot\boldsymbol{\nabla})P}{\rho_0}}_{=(\mathbf{v}\cdot\boldsymbol{\nabla})h\text{ using (1) above}}-\overbrace{\cancel{\mathbf{v}\cdot(\mathbf{v}\times\boldsymbol{\omega})}}^{\because\,\mathbf{v}\perp(\mathbf{v}\times\boldsymbol{\omega})}=0$$

$$\hookrightarrow (\mathbf{v}\cdot\boldsymbol{\nabla})\underbrace{\left(\frac{1}{2}v^2+\Phi+h\right)}_{\equiv B}\equiv(\mathbf{v}\cdot\boldsymbol{\nabla})B=0$$

$$\hookrightarrow (\mathbf{v}\cdot\boldsymbol{\nabla})B=\frac{\mathrm{d}B}{\mathrm{d}t}-\underbrace{\frac{\partial\cancel{B}}{\cancel{\partial t}}}_{\text{for steady flow}}=0\ \ \text{(here we used } \frac{\mathrm{d}}{\mathrm{d}t}=\frac{\partial}{\partial t}+\mathbf{v}\cdot\boldsymbol{\nabla}\text{)}$$

$$\therefore\ \boxed{\frac{\mathrm{d}B}{\mathrm{d}t}=0\ \Leftrightarrow\ B\equiv\frac{1}{2}v^2+\Phi+h=\text{const.}}\tag{1.50}$$

This equation states that when an ideal fluid flows steadily, the function B is conserved moving with a fluid element. This is the most elementary form of the Bernoulli theorem. Since the enthalpy h is written as $h=u+P\rho_0^{-1}$ $(H=U+PV)$, the enthalpy can be understood as the injection energy in the absence of kinetic and potential energy. When the fluid flows faster, by the conservation of the Bernoulli function, we expect a higher potential or a higher pressure without changing other variables.

1.4.4 Energy conservation

The energy conservation law may be expected with the energy flux defined as $\mathbf{F}=U\mathbf{v}$. However, in an external gravitational field, we should consider another contribution by the pressure.

$$U_{\mathrm{f}}=\rho_0(\frac{1}{2}v^2+u+\Phi)\ \text{for ideal fluid in the presence of gravity}$$

$$\Rightarrow\frac{\partial U}{\partial t}+\boldsymbol{\nabla}\cdot\mathbf{F}=0$$

$$\curvearrowright\ \mathbf{F}=(U\mathbf{v})\ \rightarrow(U\mathbf{v}+P\mathbf{v})$$

$$\therefore \boxed{\frac{\partial}{\partial t}\left[\rho_0\left(\frac{1}{2}v^2+u+\Phi\right)\right]+\boldsymbol{\nabla}\cdot\left[\rho_0\mathbf{v}\left(\frac{1}{2}v^2+h+\Phi\right)\right]=0} \tag{1.51}$$

By combining the energy, momentum, mass conservation laws and the first law of thermodynamics, we obtain

$$\frac{\mathrm{d}s}{\mathrm{d}t}=0 \text{ for an ideal fluid} \tag{1.52}$$

Now we summarize the ingredients of conservation equations for ideal fluids:

conserved quantity		Density		Flux	
Mass	$:\frac{\partial}{\partial t}$	ρ_0	$+\boldsymbol{\nabla}\cdot$	$\rho_0\mathbf{v}$	$=0$
Momentum	$:\frac{\partial}{\partial t}$	$\rho_0\mathbf{v}$	$+\boldsymbol{\nabla}\cdot$	$\mathbf{T}=\rho_0\mathbf{v}\otimes\mathbf{v}+P\mathbf{g}$	$=0$
Energy	$:\frac{\partial}{\partial t}$	$U=\left(\frac{1}{2}v^2+u+\Phi\right)\rho_0$	$+\boldsymbol{\nabla}\cdot$	$\mathbf{F}=\left(\frac{1}{2}v^2+h+\Phi\right)\rho_0\mathbf{v}$	$=0$

1.4.5 Viscosity and Navier-Stokes equation

In the section introducing a fluid, we showed its stress tensor as

$$\mathbf{T}=[\rho_0\mathbf{v}\otimes\mathbf{v}+P\mathbf{g}]-\zeta\theta\mathbf{g}-2\eta\boldsymbol{\sigma} \tag{1.53}$$

Fluids resist not the displacements of elements but the rate of them.

$$\boxed{\text{solid: } \mathbf{T}\leftarrow\boldsymbol{\nabla}\cdot\boldsymbol{\xi}=\frac{1}{3}\Theta\mathbf{g}+\boldsymbol{\Sigma}+\mathbf{R}}$$

$$\mathbf{v}=\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t}\downarrow \qquad \downarrow \quad \downarrow \quad \downarrow$$

$$\boxed{\text{fluid: } \mathbf{T}\leftarrow\boldsymbol{\nabla}\cdot\mathbf{v}=\frac{1}{3}\theta\mathbf{g}+\boldsymbol{\sigma}+\mathbf{r}}$$

where $\theta=\boldsymbol{\nabla}\cdot\mathbf{v}$

$$\sigma_{ij}=\frac{1}{2}(\nabla_j v_i+\nabla_i v_j)-\frac{1}{3}\theta g_{ij}$$
$$r_{ij}=\frac{1}{2}(\nabla_j v_i-\nabla_i v_j) \tag{1.54}$$

Then the Euler equation is written in terms of ζ and η:

$$(1) \text{ from momentum conservation: } \frac{\partial(\rho_0 \mathbf{v})}{\partial t} + \nabla \cdot \underbrace{(\rho_0 \mathbf{v} \otimes \mathbf{v} + P\mathbf{g})}_{\mathsf{T}_{\text{ideal fluid}}} = \rho_0 \vec{g}$$

$$\Downarrow$$

$$(2) \text{ Euler equation for an ideal fluid: } \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\frac{\nabla P}{\rho_0} + \vec{g}$$

(introducing viscosities)

$$\Big\Downarrow \quad \mathsf{T}_{\text{fluid}} \equiv \rho_0 \mathbf{v} \otimes \mathbf{v} + P\mathbf{g} - \zeta\theta\mathbf{g} - 2\eta\sigma \underbrace{\cancel{- k\tau}}_{\text{shear-free rotation}}$$

$$(4) \text{ Navier-Stokes equation: } \boxed{\rho_0 \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\nabla P + \rho\vec{g} + \nabla(\zeta\theta) + 2\nabla \cdot (\eta\boldsymbol{\sigma})}$$

$\downarrow$ for incompressible flows, $\theta \simeq 0$.
and since η generally varies
slowly than the shear σ,

$$\boxed{\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\frac{\nabla P}{\rho_0} + \vec{g} + \nu \nabla^2 \mathbf{v}}$$

where $\nu \equiv \dfrac{\eta}{\rho_0}$: kinematic viscosity (1.55)

The equation form in the last line is commonly quoted as the form of the Navier-Stokes equation.

In the above derivation, we used the incompressible approximation, $\theta \simeq 0$. Usually, the expansion is considered constant for a liquid which has a very large bulk modulus K. However, it is also valid even for a gas. The validity can be seen as follows.

$$(1) \text{ Euler equation: } \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{\nabla P}{\rho_0} + \underbrace{\vec{g}}_{-\nabla\Phi}$$

$$(2) \text{ dimensions: } \left[\frac{v}{T}\right] + \left[\frac{v^2}{L}\right] = \left[\frac{\delta P}{L\rho}\right] + \left[\frac{\delta \Phi}{L}\right]$$

$$\downarrow \times [L], \text{ by using } [v] = \left[\frac{L}{T}\right]$$

$$[v^2] + \quad [v^2] \quad = \left[\frac{\delta P}{\rho}\right] + \ [\delta\Phi]$$

$$\delta P \sim C^2 \delta\rho,\ C^2 = \frac{\partial P}{\partial \rho} \ : \text{(sound speed)}$$

$$\hookrightarrow \boxed{\frac{\delta\rho}{\rho} \sim \frac{v^2}{C^2} + \frac{\delta\Phi}{C^2}}$$

(3) incompressible approximation:

1. for highly subsonic fluid speeds, $v \ll C$
2. and for modest gravitational effects, $|\delta\Phi| \ll C^2$

$$\frac{\delta\rho}{\rho} \ll 1$$

$$\Downarrow$$

$$-\frac{\mathrm{d}\rho/\mathrm{d}t}{\rho} = \frac{\mathrm{d}V/\mathrm{d}t}{\mathrm{d}V} = \boxed{\nabla \cdot \mathbf{v} = \theta \simeq 0} \tag{1.56}$$

As an example of a gas, the air at atmospheric temperature has the sound speed $C \sim 300$m/s, which is very fast compared to its flow speed. As in this case, we consider most gas flows are incompressible as well as liquid.

In most fluids the magnitude of the shear stress is linear in velocity gradient. Fluids following this law are called Newtonian named after Isaac Newton, who first used the differential equation to assumed a linear relationship between shear strain rate and shear stress for fluids. Although non-Newtonian fluids are relatively common, many common fluids, such as water and air, can be assumed to be Newtonian for practical calculations under ordinary conditions.

$$\text{fluid} \begin{cases} \text{ideal} & \\ \text{Newtonian} & \text{(ex) water, air, alcohol, ...} \\ \text{Non-Newtonian} & \text{(ex) blood, tomato ketchup, shampoo, ...} \end{cases} \tag{1.57}$$

1.5 Relativistic fluid dynamics

When a fluid flows with a large speed $v = |\mathbf{v}| \approx c$, Newtonian fluid mechanics breaks down.

1.5.1 Rest-mass conservation

In the four-dimensional spacetime, we define a four velocity, which serves as the velocity vector in the four dimensional spacetime.

$$\text{four-velocity: } u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \left(\frac{\mathrm{d}x^0}{\mathrm{d}\tau}, \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau}\right)^\mu = \Big(\underbrace{\frac{\mathrm{d}t}{\mathrm{d}\tau}}_{\equiv\gamma=1/\sqrt{1-v^2}\ \text{(Lorentz factor)}}, \underbrace{\frac{\mathrm{d}t}{\mathrm{d}\tau}}\overbrace{\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}}^{\mathbf{v}}\Big) = \gamma(1, \mathbf{v})^\mu$$

$$\text{four-current: } J^\mu = \rho_0 u^\mu = \rho_0\gamma(1, \mathbf{v})^\mu = (J^0, \mathbf{J}) \tag{1.58}$$

Then the mass conservation law is generalized as

$$\frac{\partial\rho_0}{\partial t} + \boldsymbol{\nabla}\cdot(\rho_0\mathbf{v}) = 0\,, \quad \left(\frac{\mathrm{d}\rho_0}{\mathrm{d}t} + \rho_0\boldsymbol{\nabla}\cdot\mathbf{v} = 0 \text{ Lagrangian approach}\right)$$

$$\rho_0 \to \gamma\rho_0,\ \text{then } \rho_0 \to \rho_0\gamma = J^0,\ \rho_0\mathbf{v} \to \rho_0\gamma\mathbf{v} = \mathbf{J}$$

$$\Rightarrow \frac{\partial J^0}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{J} = \partial_\mu J^\mu = 0 \ \text{ in flat spacetime}$$

$$\text{(in curved spacetime, } \nabla_\mu J^\mu = \nabla_\mu(\rho_0 u^\mu) = 0) \tag{1.59}$$

This equation is written in the Lagrangian form as

rest mass conservation:

$$\boxed{\nabla_\mu(\rho_0 u^\mu) = 0}$$

$$\hookrightarrow u^\mu(\nabla_\mu\rho_0) + \rho_0\nabla_\mu u^\mu = 0$$

$$u^\mu\nabla_\mu\rho_0 = \frac{\mathrm{d}\rho_0}{\mathrm{d}\tau}$$

$$\hookrightarrow \boxed{\frac{\mathrm{d}\rho_0}{\mathrm{d}\tau} = -\rho_0\nabla_\mu u^\mu} \tag{1.60}$$

where $\mathrm{d}/\mathrm{d}\tau$ is the derivative with respect to proper time moving with the fluid. Since the rest mass conservation indicates the total rest mass conservation, we have the following relation.

$$\frac{\mathrm{d}(\rho_0 V)}{\mathrm{d}\tau} = 0 \;\rightarrow\; \frac{\mathrm{d}\rho_0}{\mathrm{d}\tau} = -\rho_0 \frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}\tau}$$

$$\frac{\mathrm{d}\rho_0}{\mathrm{d}\tau} = -\rho_0 \nabla_\mu u^\mu$$

$$\therefore \; \boxed{\nabla_\mu u^\mu = \frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}\tau}} \tag{1.61}$$

Taking the Newtonian limit of the rest-mass conservation equation, we get

Newtonian limit of the rest-mass conservation

$$\frac{\mathrm{d}\rho_0}{\mathrm{d}\tau} = -\rho_0 \nabla_\mu u^\mu \xrightarrow[\tau\to t,\ \gamma\to 1,\ u^\mu\to(1,v^i)]{\text{non-relativistic limit}} \frac{\mathrm{d}\rho_0}{\mathrm{d}t} + \rho_0 \boldsymbol{\nabla}\cdot\mathbf{v} = 0 \tag{1.62}$$

The four-dimensional conservation law indicates that when a charge or a particle passes through a past spacelike surface, the same charge or particle exits the 4-dimensional region $\mathcal{V}_4$ through a future spacelike boundary.

$$\begin{aligned}
\int_{\mathcal{V}} \nabla_\mu J^\mu d\Sigma &= \int_{\partial\mathcal{V}} J^\mu d\Sigma_\mu \\
&= \int_{\partial\mathcal{V}_{\text{future}}} J^\mu d\Sigma_\mu - \int_{\partial\mathcal{V}_{\text{past}}} J^\mu d\Sigma_\mu + \cancel{\int_{\partial\mathcal{V}_{\text{infinity}}} J^\mu d\Sigma_\mu} \\
&= 0
\end{aligned}$$

$$\therefore \int_{\partial\mathcal{V}_{\text{future}}} J^\mu d\Sigma_\mu = \int_{\partial\mathcal{V}_{\text{past}}} J^\mu d\Sigma_\mu \tag{1.63}$$

1.5.2 Energy conservation

In the Newtonian fluid mechanics we have the following conservation equation for energy:

$$\frac{\partial}{\partial t}\left[\rho_0\left(\frac{1}{2}v^2 + u + \Phi\right)\right] + \boldsymbol{\nabla}\cdot\left[\rho_0\mathbf{v}\left(\frac{1}{2}v^2 + h + \Phi\right)\right] = 0 \tag{1.64}$$

The relativistic version can be derived from the thermodynamics first law:

$$\mathrm{d}E = T\mathrm{d}S - P\mathrm{d}V$$

$$\hookrightarrow \frac{\mathrm{d}(\rho V)}{\mathrm{d}\tau} = T\frac{\mathrm{d}(\rho_0 V s)}{\mathrm{d}\tau} - P\frac{\mathrm{d}V}{\mathrm{d}\tau}$$

$$
\begin{aligned}
&= T \underbrace{\cancel{\frac{\mathrm{d}(\rho_0 V)}{\mathrm{d}\tau}}}_{\text{rest-mass conservation}} s + T(\rho_0 V) \underbrace{\cancel{\frac{\mathrm{d}s}{\mathrm{d}\tau}}}_{\text{for adiabaticity of fluids}} - P\frac{\mathrm{d}V}{\mathrm{d}\tau} \\
&= -P\frac{\mathrm{d}V}{\mathrm{d}\tau}
\end{aligned}
$$

$$
\hookrightarrow \boxed{\frac{\mathrm{d}\rho}{\mathrm{d}\tau} = -(\rho+P)\frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}\tau} = -(\rho+P)\nabla_\mu u^\mu} \tag{1.65}
$$

In the Newtonian limit, we obtain the rest-mass conservation equation.

$$
\underbrace{\frac{\mathrm{d}\rho}{\mathrm{d}\tau}}_{\approx \frac{\mathrm{d}\rho_0}{\mathrm{d}t}} = -\underbrace{(\rho+P)}_{\to\rho_0}\underbrace{\nabla_\mu u^\mu}_{=\nabla_t u^t + \nabla_i u^i \to \boldsymbol{\nabla}\cdot\mathbf{v}} \xrightarrow{\text{non-relativistic limit}} \frac{\mathrm{d}\rho_0}{\mathrm{d}t} = -\rho_0(\boldsymbol{\nabla}\cdot\mathbf{v}) \tag{1.66}
$$

1.5.3 Conservation laws in terms of the stress-energy tensor

In the four-dimensional spacetime, the stress tensor of the ideal (perfect) fluid defined in the Newtonian fluid mechanics is extended in terms of four-dimensional quantities.

$\mathbf{T} = \rho_0 \mathbf{v}\otimes\mathbf{v} + P\mathbf{g}$ (ideal fluid in Newtonian fluid mechanics)

$\downarrow \quad \downarrow \quad \downarrow \quad\quad \downarrow$

$\boldsymbol{T} = \rho\boldsymbol{u}\otimes\boldsymbol{u} + P\boldsymbol{P}$ (ideal fluid in 4D spacetime)

where $\mathbf{T} = T^{ij}\partial_i\partial_j$: stress tensor,

$\boldsymbol{T} = T^{\mu\nu}\partial_\mu\partial_\nu$: stress-energy (energy-momentum) tensor

ρ_0 : rest-mass density

$\boldsymbol{u} = u^\mu\partial_\mu$: four-velocity

$\boldsymbol{P} = P^{\mu\nu}\partial_\mu\partial_\nu$: tensor projecting on a spacelike hypersurface

with $\boldsymbol{u}\otimes\boldsymbol{u},\ \boldsymbol{P}$ we have $g^{\mu\nu} = -u^\mu u^\nu + P^{\mu\nu}\ \cdots (P)$ (1.67)

Since the comoving observer with the fluid will observe an isotropic distribution of particles, the only quantity coming into the stress-energy tensor is an invariant scalar under the spacial rotation. This is why the four-dimensional stress-energy tensor is defined in terms of the energy along the time-like direction and the pressure on the spacelike hypersurface. By using the projection tensor relation (P) and introducing thermodynamics properties, the perfect fluid stress-energy tensor in the rest frame of the observer is written by

$$
\begin{aligned}
T^{\mu\nu} &= \rho u^\mu u^\nu + P \underbrace{P^{\mu\nu}}_{=g^{\mu\nu}+u^\mu u^\nu} \\
\hookrightarrow\ & \boxed{T^{\mu\nu} = (\rho + P) u^\mu u^\nu + P g^{\mu\nu}} \\
&= [\rho_0 (1+u) + P] u^\mu u^\nu + P g^{\mu\nu} \\
&= \rho_0 \underbrace{(1 + u + \frac{P}{\rho_0})}_{=1+u+Pv=h} u^\mu u^\nu + P g^{\mu\nu} \\
\therefore\ & \boxed{T^{\mu\nu} = \rho_0 h u^\mu u^\nu + P g^{\mu\nu}}
\end{aligned} \tag{1.68}
$$

where ρ_0, ρ and P are the rest mass, the total energy density and the pressure, respectively. The specific volume v, internal energy u and the specific enthalpy h are introduced in the third and fourth line.

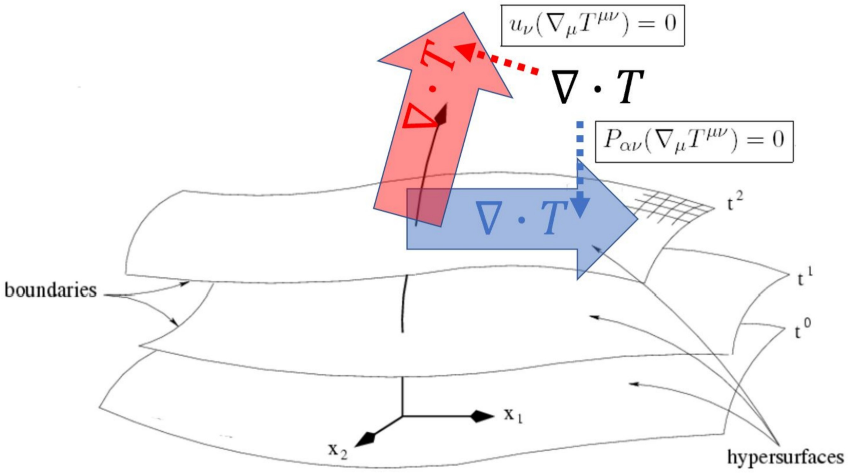

Figure 1.12: Projections of the relativistic conservation law: Along the timelike four velocity, we get the energy conservation law. On the spacelike hypersurface, the projected conservation law leads to the relativistic Euler equation.

The conservation laws of energy and momentum are described by the timelike and spacelike components of the continuity equation $\nabla_\mu T^{\mu\nu} = 0$:

(1) **energy conservation:**

$$\boxed{u_\nu(\nabla_\mu T^{\mu\nu}) = 0}$$

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + Pg^{\mu\nu}$$

$$u_\nu\{\nabla_\mu[(\rho+P)u^\mu u^\nu + Pg^{\mu\nu}]\} = u_\nu\{\nabla_\mu(\rho+P)u^\mu u^\nu + (\rho+P)(\nabla_\mu u^\mu)u^\nu$$
$$+ (\rho+P)u^\mu(\nabla_\mu u^\nu) + (\nabla_\mu P)g^{\mu\nu} + P\cancel{\nabla_\mu g^{\mu\nu}}]\}$$

$u_\mu u^\mu = -1$ for massive particles

$$= -\nabla_\mu(\rho + \cancel{P})u^\mu - (\rho+P)(\nabla_\mu u^\mu)$$
$$+ (\rho+P)u^\mu \cancel{\underbrace{u_\nu(\nabla_\mu u^\nu)}_{=0 \because \nabla_\mu(u_\nu u^\nu)=0} + (\nabla_\mu P)u^\mu}$$

$$= -\underbrace{(\nabla_\mu \rho) u^\mu}_{=\frac{\mathrm{d}\rho}{\mathrm{d}\tau}} - (\rho + P)(\nabla_\mu u^\mu) = 0$$

$$\therefore \boxed{\frac{\mathrm{d}\rho}{\mathrm{d}\tau} = -(\rho + P)\nabla_\mu u^\mu}$$

(2) **momentum conservation:**

$$\boxed{P_{\alpha\nu}(\nabla_\mu T^{\mu\nu}) = 0}, \quad (P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu)$$

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + P g^{\mu\nu}, \quad P_{\mu\nu}u^\nu = 0$$

$$P_{\alpha\nu}\{\nabla_\mu[(\rho + P)u^\mu u^\nu + P g^{\mu\nu}]\}$$

$$= P_{\alpha\nu}\{\nabla_\mu[(\rho + P)u^\mu]u^\nu + (\rho + P)u^\mu(\nabla_\mu u^\nu) + (\nabla_\mu P)g^{\mu\nu} + P\nabla_\mu g^{\mu\nu}\}$$

$$= \nabla_\mu[(\rho + P)u^\mu]\overbrace{\cancel{P_{\alpha\nu}u^\nu}}^{=0 \text{ by zero projection of } u^\mu} + (\rho + P)u^\mu \underbrace{P_{\alpha\nu}(\nabla_\mu u^\nu)}_{=(g_{\alpha\nu} + u_\alpha u_\nu)(\nabla_\mu u^\nu) = \nabla_\mu u_\alpha \ \because\ u_\nu \nabla_\mu u^\nu = 0} + (\nabla_\mu P)P_{\alpha\nu}g^{\mu\nu} + P_{\alpha\nu}P\overbrace{\cancel{\nabla_\mu g^{\mu\nu}}}^{=0 \text{ by metric compatibility}}]\}$$

$$= (\rho + P)\underbrace{u^\mu \nabla_\mu u_\alpha}_{=\frac{\mathrm{d}u_\alpha}{\mathrm{d}\tau}} + (\nabla_\mu P)P_\alpha{}^\mu = 0$$

$$\therefore \boxed{(\rho + P)\frac{\mathrm{d}u^\alpha}{\mathrm{d}\tau} = -P^{\alpha\mu}\nabla_\mu P} \tag{1.69}$$

1.5.4 Relativistic Euler equation

From the relativistic momentum conservation law of fluids, we obtain the relativistic Euler equation:

relativistic Euler equation:

$$(\rho + P)\frac{\mathrm{d}u^\alpha}{\mathrm{d}\tau} = -P^{\alpha\mu}\nabla_\mu P$$

$$P^{\alpha\mu} = g^{\alpha\mu} + u^\alpha u^\mu$$

$$(\rho + P)\frac{\mathrm{d}u^\alpha}{\mathrm{d}\tau} = -(g^{\alpha\mu} + u^\alpha u^\mu)\nabla_\mu P$$

$$= -\nabla^\alpha P - u^\alpha \underbrace{u^\mu \nabla_\mu P}_{=\frac{\mathrm{d}P}{\mathrm{d}\tau}}$$

$$= -\nabla^\alpha P - u^\alpha \frac{\mathrm{d}P}{\mathrm{d}\tau}$$

$$\hookrightarrow \boxed{(\rho + P)\frac{\mathrm{d}u^\alpha}{\mathrm{d}\tau} = -\nabla^\alpha P - u^\alpha \frac{\mathrm{d}P}{\mathrm{d}\tau}} \tag{1.70}$$

In the low velocity limit ($u^i \ll 1$), that is, $\rho \gg P$ and $\rho \approx \rho_0$, we again get the Euler equation in the Newtonian fluid mechanics:

$$(\rho + P)\frac{\mathrm{d}u^\alpha}{\mathrm{d}\tau} = -\nabla^\alpha P - u^\alpha \frac{\mathrm{d}P}{\mathrm{d}\tau}$$

$i = 1, 2, 3$ component

$$\underbrace{(\rho + P)}_{\to \rho_0} \underbrace{\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\tau}}_{\tau \approx t} = -\vec{\nabla} P - \underbrace{\mathbf{u}}_{\ll 1} \frac{\mathrm{d}P}{\mathrm{d}\tau} \xrightarrow{\text{non-relativistic limit}} \rho_0 \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = -\boldsymbol{\nabla} P \tag{1.71}$$

Now we have all equations for all unknowns in the relativistic fluid dynamics.

(1) eight unknowns: $(\rho_0,\ \rho,\ P,\ s,\ \boldsymbol{u})$

(2) eight differential equations

$$\begin{cases} \text{(1) rest-mass conservation} & \frac{\mathrm{d}\rho_0}{\mathrm{d}\tau} = -\rho_0 \nabla_\mu u^\mu \ \to \ \text{one DE} \\ \text{(2) energy conservation} & \frac{\mathrm{d}\rho}{\mathrm{d}\tau} = -(\rho + P)\nabla_\mu u^\mu \ \to \ \text{one DE} \\ \text{(3) first law of TD with (1),(2)} & \frac{\mathrm{d}s}{\mathrm{d}\tau} = 0 \ \to \ \text{one DE} \\ \text{(4) relativistic Euler equation} & (\rho + P)\frac{\mathrm{d}u^\alpha}{\mathrm{d}\tau} = -\nabla^\alpha P - u^\alpha \frac{\mathrm{d}P}{\mathrm{d}\tau} \ \to \text{four DE} \\ \text{(5) equation of state} & P = P(\rho_0, s) \end{cases} \tag{1.72}$$

In summary, here we show all the conservation laws we have been dealt with in the fluid dynamics:

conserved quantities	Newtonian fluid dynamics	relativistic fluid dynamics
mass	$\begin{cases} \text{E: } \dfrac{\partial \rho_0}{\partial t} + \boldsymbol{\nabla}\cdot(\rho_0 \mathbf{v}) = 0 \\ \text{L: } \dfrac{\mathrm{d}\rho_0}{\mathrm{d}t} + \rho_0 \boldsymbol{\nabla}\cdot\mathbf{v} = 0 \end{cases}$	$\frac{\mathrm{d}(\rho_0 V)}{\mathrm{d}\tau} = 0$ $\hookrightarrow \boxed{\dfrac{\mathrm{d}\rho_0}{\mathrm{d}\tau} = -\rho_0 \nabla_\mu u^\mu} = -\rho_0 \frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}\tau}$
energy	$\dfrac{\partial U}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{F} = 0$ $U = (\dfrac{1}{2}v^2 + u + \Phi)\rho_0$ $\mathbf{F} = \left(\dfrac{1}{2}v^2 + h + \Phi\right)\rho_0\mathbf{v}$	$u_\nu(\nabla_\mu T^{\mu\nu}) = 0$ $\hookrightarrow \boxed{\dfrac{\mathrm{d}\rho}{\mathrm{d}\tau} = -(\rho+P)\nabla_\mu u^\mu}$ ↑ adiabatic $\left(\frac{\mathrm{d}(\rho V)}{\mathrm{d}\tau} = T\frac{\mathrm{d}(\rho_0 V s)}{\mathrm{d}\tau} - P\frac{\mathrm{d}V}{\mathrm{d}\tau}\right)$
momentum	$\frac{\partial(\rho_0\mathbf{v})}{\partial t} + \boldsymbol{\nabla}\cdot\underbrace{(\rho_0\mathbf{v}\otimes\mathbf{v} + P\mathbf{g})}_{\mathbf{T}_{\text{ideal fluid}}} = \rho_0\vec{g}$	$P_{\alpha\nu}(\nabla_\mu T^{\mu\nu}) = 0$ $\hookrightarrow \boxed{(\rho+P)\dfrac{\mathrm{d}u^\alpha}{\mathrm{d}\tau} = -P^{\alpha\mu}\nabla_\mu P}$
Euler equation	$\begin{cases} \frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{\boldsymbol{\nabla}P}{\rho_0} + \vec{g} \\ \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\frac{\boldsymbol{\nabla}P}{\rho_0} + \vec{g} \end{cases}$	↓ $\boxed{(\rho+P)\dfrac{\mathrm{d}u^\alpha}{\mathrm{d}\tau} = -\nabla^\alpha P - u^\alpha\dfrac{\mathrm{d}P}{\mathrm{d}\tau}}$
Bernoulli theorem	$\dfrac{\partial\mathbf{v}}{\partial t} - \boldsymbol{\nabla}\left(\dfrac{1}{2}v^2 + \Phi\right)$ $+ \dfrac{\boldsymbol{\nabla}P}{\rho_0} - \mathbf{v}\times\boldsymbol{\omega} = 0$ for steady flows, $\frac{\mathrm{d}B}{\mathrm{d}t} = 0$ $\Leftrightarrow\ B \equiv \frac{1}{2}v^2 + \Phi + h = \text{const.}$	$\mathcal{L}_{\boldsymbol{u}}\left(hu_\beta t^\beta\right) = \mathcal{L}_{\boldsymbol{u}}\left(\frac{\rho+P}{\rho_0}u_\beta t^\beta\right) = 0$
Navier-Stokes equation	$\rho_0\dfrac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\boldsymbol{\nabla}P + \rho\vec{g}$ $+ \boldsymbol{\nabla}(\zeta\theta) + 2\boldsymbol{\nabla}\cdot(\eta\boldsymbol{\sigma})$ ↓ incompressible ↓ slowly varying η $\dfrac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\dfrac{\boldsymbol{\nabla}P}{\rho_0} + \vec{g} + \nu\nabla^2\mathbf{v}$	....