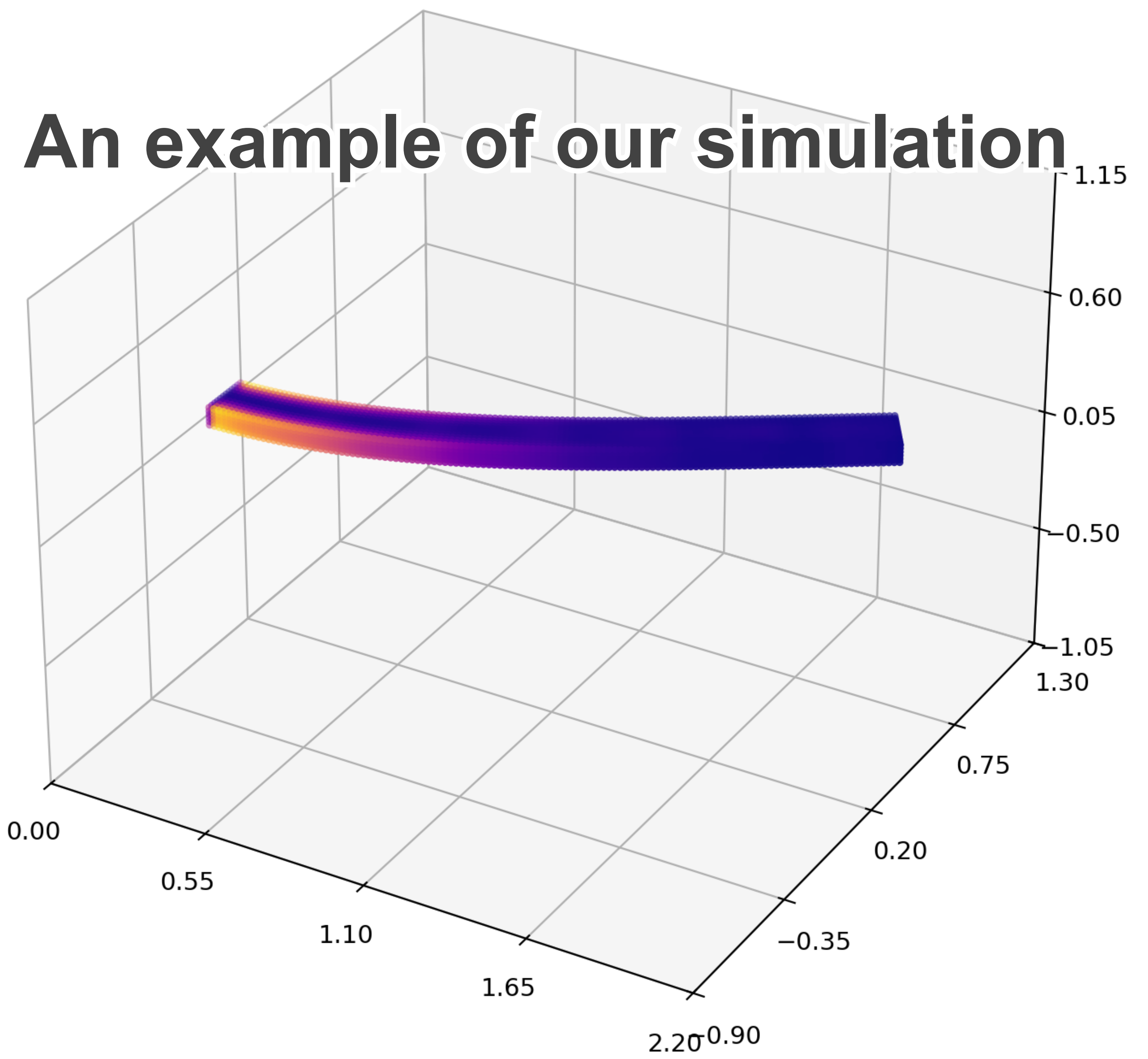
An Extension of Smoothed Particle Hydrodynamics to Elastic Dynamics only with Central Force

Take-Home Messages:

- ▶ Spring-like force is incorporated into SPH
- ▶ Significantly reduces computational costs
- ▶ Achieves excellent conservation properties



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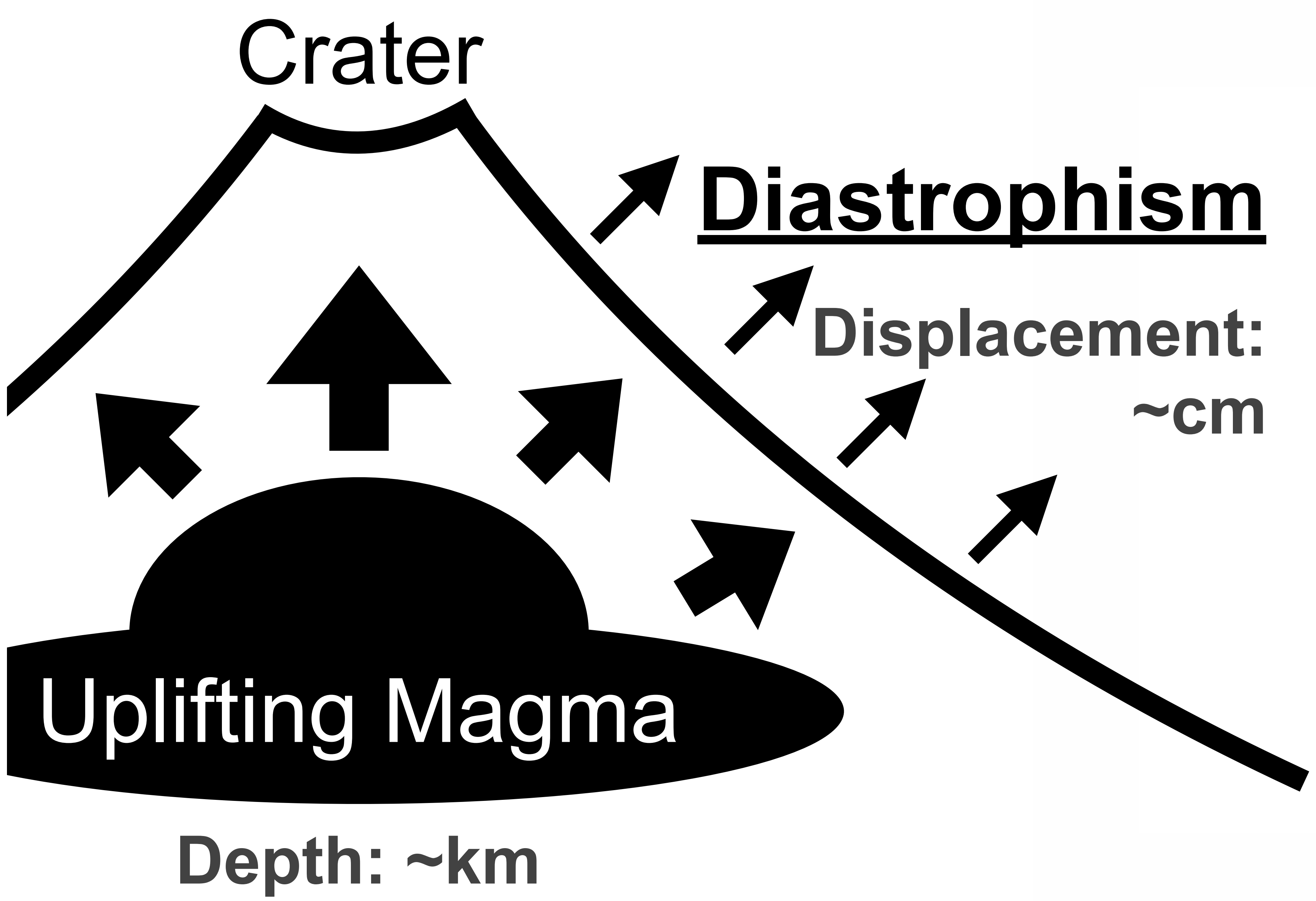
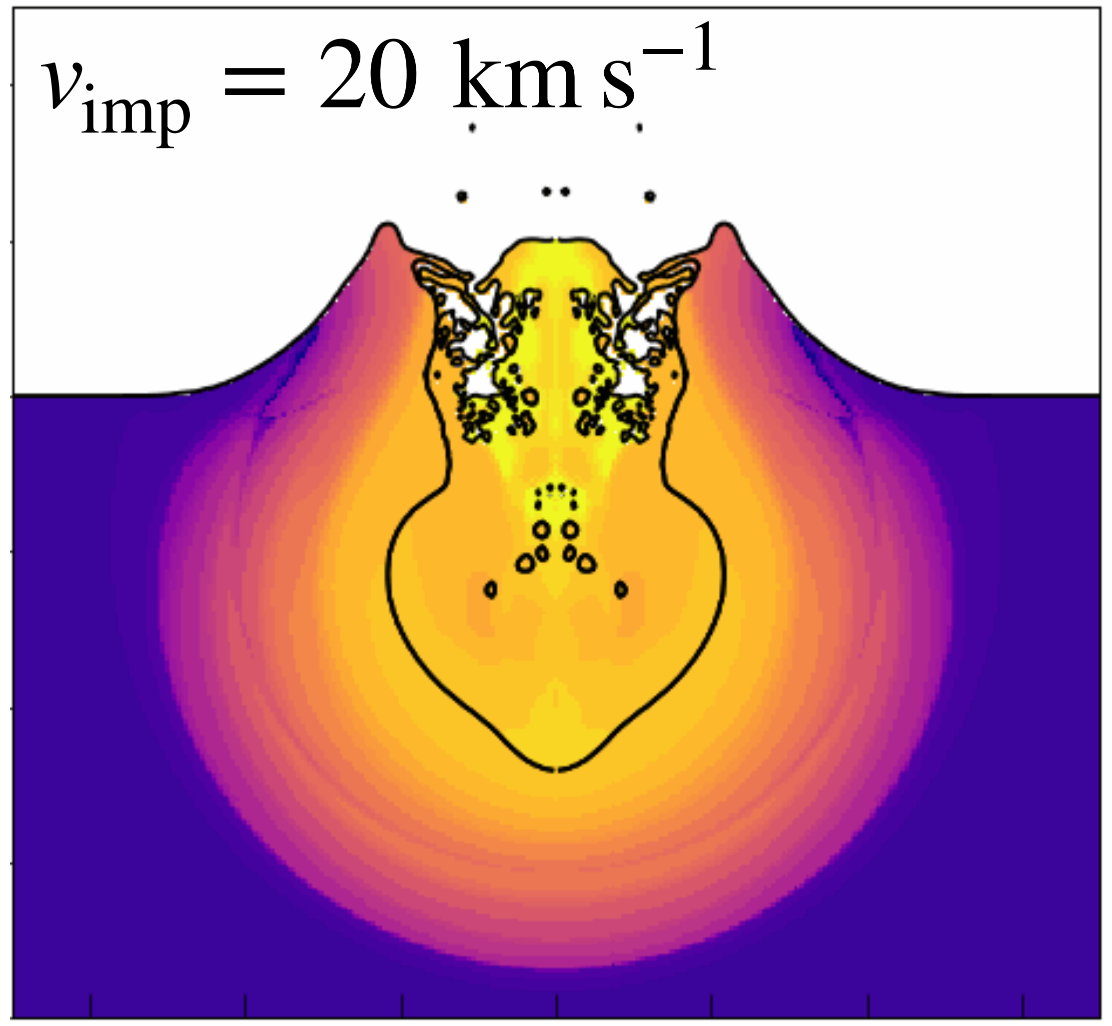
Computational Elastic Dynamics

- Elastic dynamics describes the mechanical behavior of **solid bodies**
- **Numerical simulations** are widely used to study various phenomena that cannot be studied by experiments

Examples of simulation:

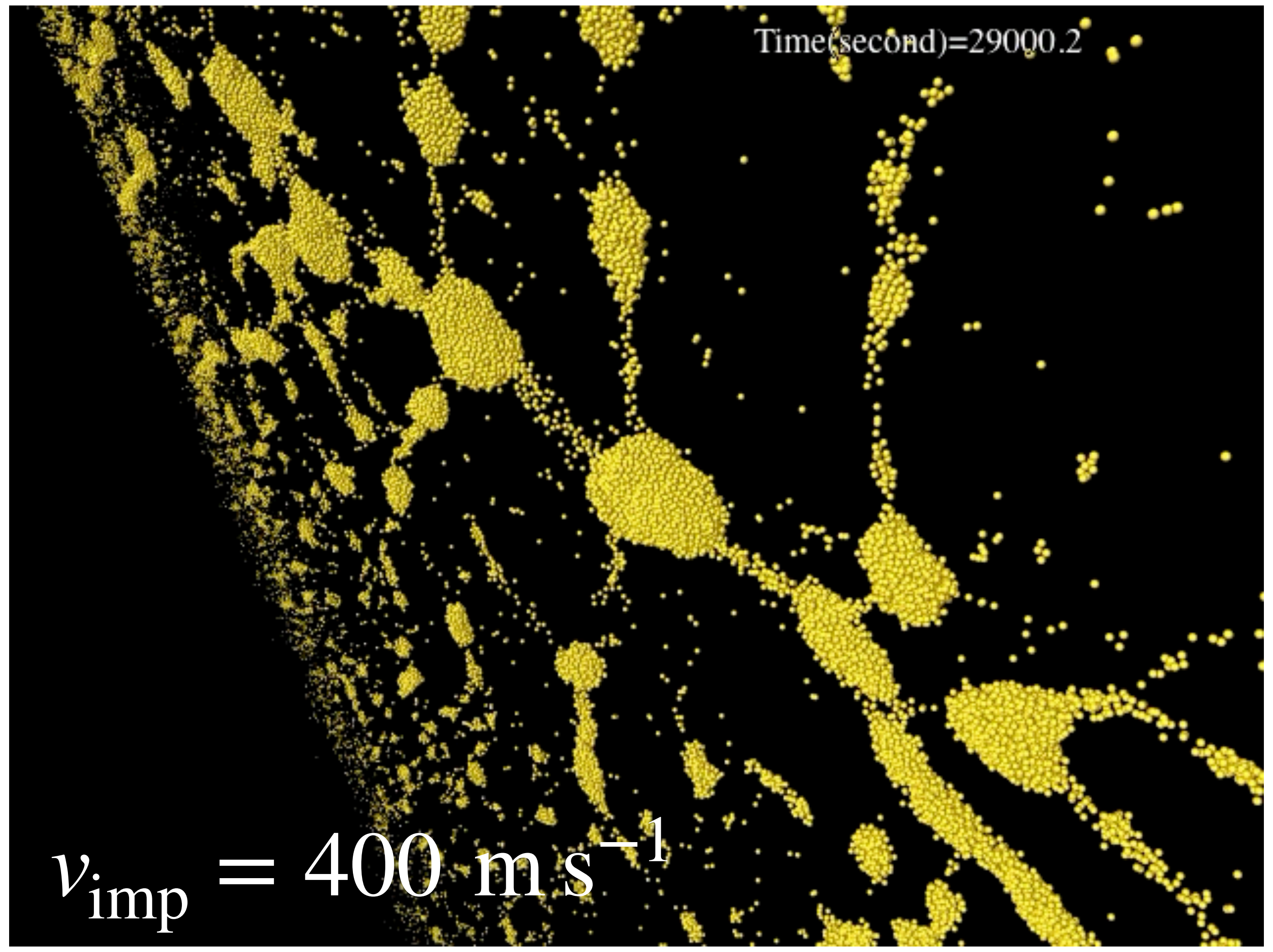


Meteorite impact



[Miyayama et al,. 2024]

Planetesimal collision



[Sugiura et al., 2018]

Existing Numerical Methods for Elastic Dynamics

SPH Extended to "Elastic Dynamics"

- Smoothed particle hydrodynamics is extended to elastic dynamics

[Libersky et al., 1991; Jutzi et al., 2015; Sugiura et al., 2017]

Basic Eqs. before discretization:

- Mass $$\frac{d\rho}{dt} = -\rho \frac{\partial}{\partial x^\alpha} v^\alpha$$
- Momentum $$\frac{dv^\alpha}{dt} = \frac{1}{\rho} \frac{\partial}{\partial x^\beta} \sigma^{\alpha\beta}$$
- Energy $$\frac{du}{dt} = \frac{1}{\rho} \sigma^{\alpha\beta} \frac{\partial}{\partial x^\beta} v^\alpha$$
- Eq. of state $$P = P(\rho, u)$$
- **Eqs. of deviatoric stress tensor**

$$\frac{dS^{\alpha\beta}}{dt} = 2\mu \left(\dot{\epsilon}^{\alpha\beta} - \frac{1}{3} \dot{\epsilon}^{\gamma\gamma} \right) + S^{\alpha\gamma} R^{\beta\gamma} + S^{\beta\gamma} R^{\alpha\gamma}$$

Stress tensor

$$\sigma^{\alpha\beta} = -P\delta^{\alpha\beta} + S^{\alpha\beta}$$

P :**Pressure** that is negative for attraction

$S^{\alpha\beta}$:**Deviatoric stress** that vanishes for fluid

In order to maintain covariance in the equations, the deviatoric stress tensor must be time integrated

μ :Shear modulus $R^{\alpha\beta}$:Rotation rate $\dot{\epsilon}^{\alpha\beta}$:Strain rate

Problems of Existing Methods

① Comptational cost

Equations of deviatoric stress

Five simultaneous differential equations

$$\frac{dS^{\alpha\beta}}{dt} = 2\mu\left(\dot{\epsilon}^{\alpha\beta} - \frac{1}{3}\dot{\epsilon}^{\gamma\gamma}\right) + S^{\alpha\gamma}R^{\beta\gamma} + S^{\beta\gamma}R^{\alpha\gamma}$$

Expanding the summations results in no fewer than 20 terms!

Redundant & large computational costs

② Angular momentum

- Interaction by **non-central force**
- Particles' **inertial moment are neglected**

Total angular momentum is not strictly conserved

Need to construct a new method without a time integration of stress and only with central forces

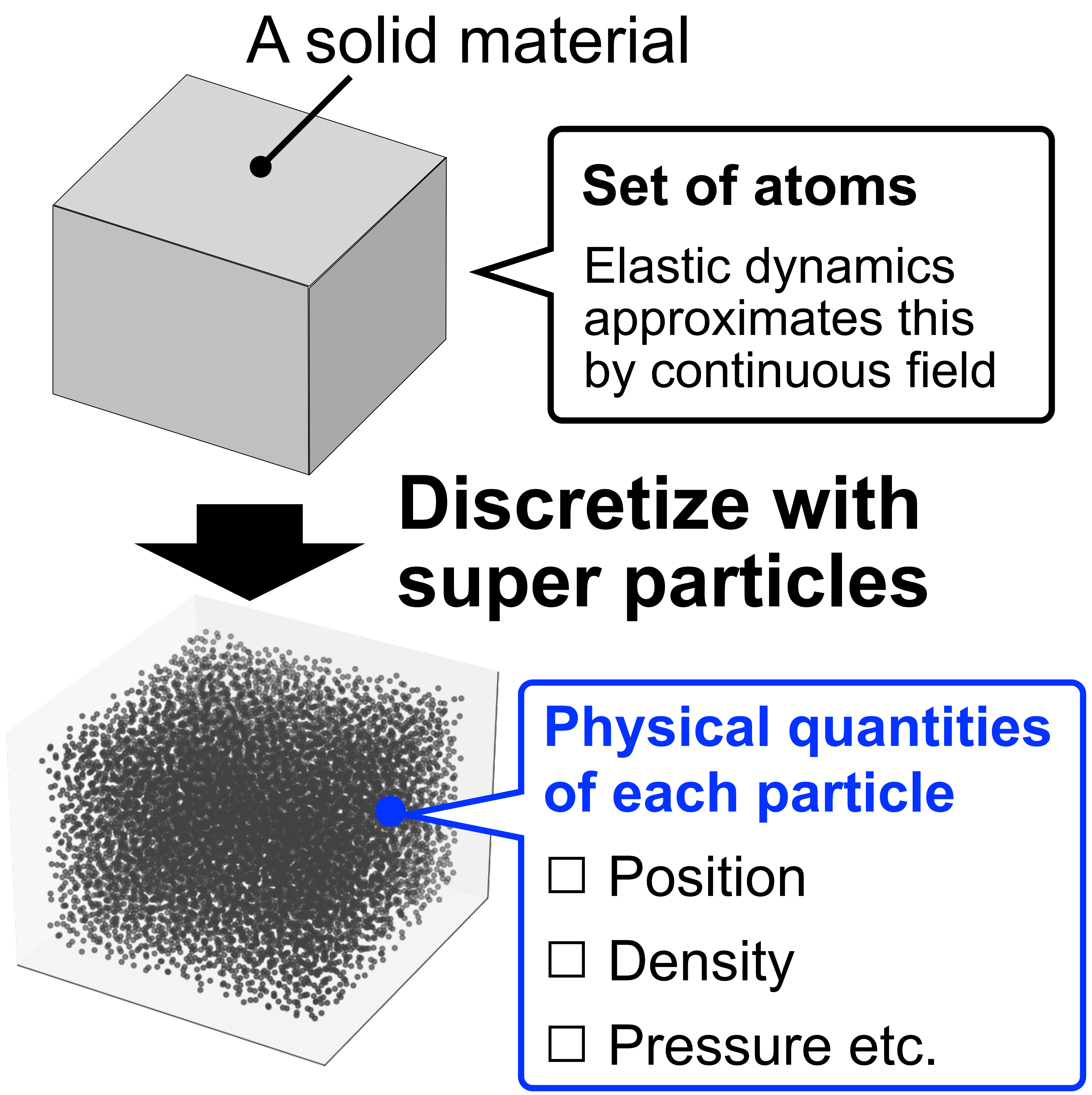
A New Formulation:
SPH with Spring Force

Computing stress tensors without time integration

The Basis of Spring SPH

▶ Basic concepts

- ☐ Discretizes continuous elastic field with particles
- ☐ Extends **SPH formulation** to elastic dynamics introducing **spring force** into inter-particle interaction



▶ Properties

- ☐ Suitable for describing **dynamic phenomena**
- ☐ Represents the non-diagonal components of the stress tensor **only with central force**
- ☐ Conserves **angular momentum** rigorously
- ☐ Significantly **reduces computational costs**

Extended Equation of Motion

Equation of motion:

k_{spr} :Spring-to-SPH ratio

$$m_i \frac{d\dot{\boldsymbol{x}}_i}{dt} = -\frac{1}{k_{\mathrm{spr}}+1} \sum_j \left[\mathcal{P} V_{\mathrm{p}}^2\right]_{ij} \frac{\partial}{\partial \boldsymbol{x}_i} W(|\boldsymbol{x}_i - \boldsymbol{x}_j|, h) - \frac{k_{\mathrm{spr}}}{k_{\mathrm{spr}}+1} \sum_j \left[\mathcal{P}^S V_{\mathrm{p}}^2\right]_{ij} \frac{\partial}{\partial \boldsymbol{x}_i} W^S(|\boldsymbol{x}_i - \boldsymbol{x}_j|, h)$$

Normal SPH term

$k_{\mathrm{spr}} \to 0$

EoS: $\mathcal{P}_i = \mathcal{C}_{\mathrm{s}}^2 (\rho_i - \rho_0)$

$\mathcal{C}_{\mathrm{s}}$:SPH bulk sound speed

ρ_0 :Initial density

Kernel: $W(r, h)$ **Not specified**

Enforces nearly incompressible

Additional term analogous to spring force

$k_{\mathrm{spr}} \to \infty$

EoS: $\mathcal{P}_i^S = \mathcal{C}_{\mathrm{s}}^2 \rho_i$

Spring kernel: $W^S(r,h) = a(|\boldsymbol{x}_i - \boldsymbol{x}_j| - L_{ij})^2 \, h^{-2} \, W(r,h)$

Initial distance: $L_{ij} = |\boldsymbol{x}_{i,0} - \boldsymbol{x}_{j,0}|$

Kernel normalization factor: $a = \left(\frac{3}{2} - \frac{4}{\sqrt{\pi}} \frac{L_{ij}}{h} + \frac{L_{ij}^2}{h^2} \right)^{-1}$

for Gaussian kernel

Note: No requirement of pressure gradient terms of a specific SPH formulation

"Spring" Term in Equation of Motion

Equation of motion:

$k_{\rm spr}$:Spring-to-SPH ratio

$$m_i \frac{d\dot{\boldsymbol{x}}_i}{dt} = -\frac{1}{k_{\rm spr}+1}\sum_j \left[\mathcal{P}V_{\rm p}^2\right]_{ij} \frac{\partial}{\partial \boldsymbol{x}_i} W(|\boldsymbol{x}_i - \boldsymbol{x}_j|, h) - \frac{k_{\rm spr}}{k_{\rm spr}+1}\sum_j \left[\mathcal{P}^S V_{\rm p}^2\right]_{ij} \frac{\partial}{\partial \boldsymbol{x}_i} W^S(|\boldsymbol{x}_i - \boldsymbol{x}_j|, h)$$

Why is the second term "spring"?

Gradient of the **spring kernel**:

$$\frac{\partial}{\partial \boldsymbol{x}_i} W^S = 2a(|\boldsymbol{x}_i - \boldsymbol{x}_j| - L_{ij}) \frac{\boldsymbol{x}_i - \boldsymbol{x}_j}{|\boldsymbol{x}_i - \boldsymbol{x}_j|} h^{-2} W + a(|\boldsymbol{x}_i - \boldsymbol{x}_j| - L_{ij})^2 h^{-2} \frac{\partial}{\partial \boldsymbol{x}_i} W$$

The second term is negligible for small deformation: $||\boldsymbol{x}_i - \boldsymbol{x}_j| - L_{ij}| \ll h$

Generalized **spring force** in 3D:

$$\boldsymbol{f}_{ij} = -k(|\boldsymbol{x}_i - \boldsymbol{x}_j| - L_{ij}) \frac{\boldsymbol{x}_i - \boldsymbol{x}_j}{|\boldsymbol{x}_i - \boldsymbol{x}_j|}$$

k :spring constant

Our formulation incorporates a spring force into the pressure gradient force of SPH

Basic Equations

Equation of motion:

$k_{\rm spr}$:Spring-to-SPH ratio

$$m_i \frac{d\dot{\boldsymbol{x}}_i}{dt} = -\frac{1}{k_{\rm spr}+1}\sum_j \left[\mathcal{P}V_{\rm p}^2\right]_{ij} \frac{\partial}{\partial \boldsymbol{x}_i} W(|\boldsymbol{x}_i - \boldsymbol{x}_j|, h) - \frac{k_{\rm spr}}{k_{\rm spr}+1}\sum_j \left[\mathcal{P}^S V_{\rm p}^2\right]_{ij} \frac{\partial}{\partial \boldsymbol{x}_i} W^S(|\boldsymbol{x}_i - \boldsymbol{x}_j|, h)$$

Density: **(for Volume-based SPH)**

$$\rho_i = m_i \sum_j W(|\boldsymbol{x}_i - \boldsymbol{x}_j|, h)$$

Equation of state: $P = P(\rho, u)$

Energy equation:

$$m_i \frac{du_i}{dt} = \frac{1}{k_{\rm spr}+1} \cdot \frac{1}{2} \sum_j \left[\mathcal{P}V_{\rm p}^2\right]_{ij} (\dot{\boldsymbol{x}}_i - \dot{\boldsymbol{x}}_j) \cdot \frac{\partial}{\partial \boldsymbol{x}_i} W(|\boldsymbol{x}_i - \boldsymbol{x}_j|, h) + \frac{k_{\rm spr}}{k_{\rm spr}+1} \cdot \frac{1}{2} \sum_j \left[\mathcal{P}^S V_{\rm p}^2\right]_{ij} (\dot{\boldsymbol{x}}_i - \dot{\boldsymbol{x}}_j) \cdot \frac{\partial}{\partial \boldsymbol{x}_i} W^S(|\boldsymbol{x}_i - \boldsymbol{x}_j|, h)$$

▶ For now, **Volume-based SPH** [Seno et al., in prep.] is employed for the SPH formulation

$$\left[PV_{\rm p}{}^2\right]_{ij} = P_i V_{{\rm p},i}^2 + P_j V_{{\rm p},j}^2 \qquad V_{{\rm p},i} = \frac{m_i}{\rho_i}$$

Since Volume-based SPH can describe the density discontinuity accurately, it's suitable for **materials exposed to the vacuum**

Conservation Properties

Basic equations discretized in time:

Momentum: $$m_i \frac{\Delta \dot{\boldsymbol{x}}_i}{\Delta t} = -\frac{1}{k_{\rm spr}+1}\sum_j \left[\mathcal{P}V_{\rm p}^2\right]_{ij} \frac{\partial}{\partial \boldsymbol{x}_i} W(|\boldsymbol{x}_i - \boldsymbol{x}_j|, h) - \frac{k_{\rm spr}}{k_{\rm spr}+1}\sum_j \left[\mathcal{P}^S V_{\rm p}^2\right]_{ij} \frac{\partial}{\partial \boldsymbol{x}_i} W^S(|\boldsymbol{x}_i - \boldsymbol{x}_j|, h)$$

Energy: $$m_i \frac{\Delta u_i}{\Delta t} = \frac{1}{k_{\rm spr}+1}\cdot\frac{1}{2}\sum_j \left[\mathcal{P}V_{\rm p}^2\right]_{ij} (\dot{\boldsymbol{x}}_i^* - \dot{\boldsymbol{x}}_j)\cdot \frac{\partial}{\partial \boldsymbol{x}_i} W(|\boldsymbol{x}_i - \boldsymbol{x}_j|, h) + \frac{k_{\rm spr}}{k_{\rm spr}+1}\cdot\frac{1}{2}\sum_j \left[\mathcal{P}^S V_{\rm p}^2\right]_{ij} (\dot{\boldsymbol{x}}_i^* - \dot{\boldsymbol{x}}_j)\cdot \frac{\partial}{\partial \boldsymbol{x}_i} W^S(|\boldsymbol{x}_i - \boldsymbol{x}_j|, h)$$

Next velocity:

$$\dot{\boldsymbol{x}}_i^* = \dot{\boldsymbol{x}}_i + \Delta \dot{\boldsymbol{x}}_i$$

▶ Time variatoin of conservative quantities:

Achieved only in our scheme!

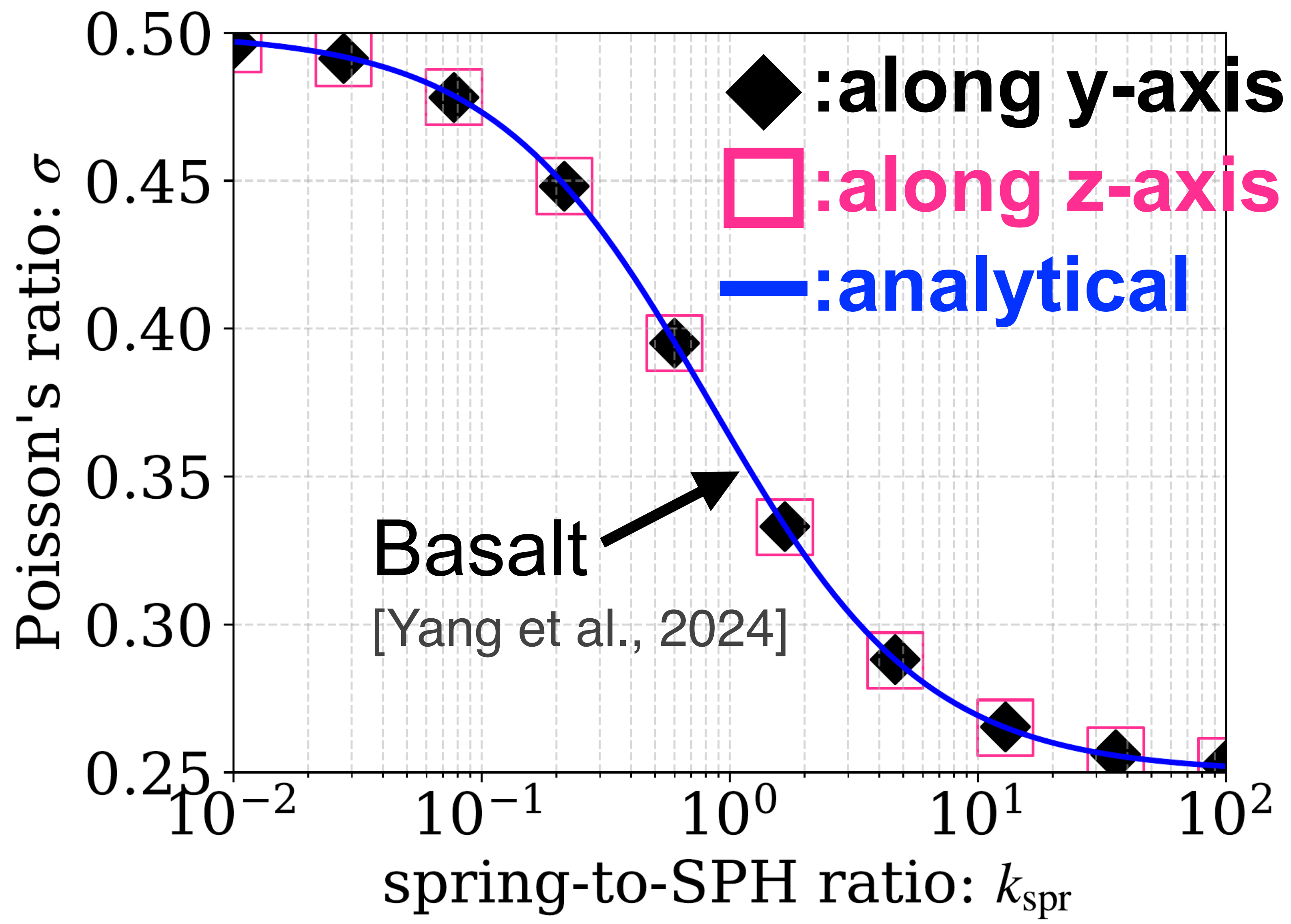
- Mass $$\frac{d}{dt}\sum_i m_i = 0$$
- **Angular momentum** $$\Delta\left(\sum_i \boldsymbol{x}_i \times m_i \dot{\boldsymbol{x}}_i\right) = \boldsymbol{0}$$
- Mometum $$\Delta\left(\sum_i m_i \dot{\boldsymbol{x}}_i\right) = \boldsymbol{0}$$
- Energy $$\Delta\left(\sum_i m_i\left[\frac{1}{2}|\dot{\boldsymbol{x}}_i|^2 + u_i\right]\right) = 0$$

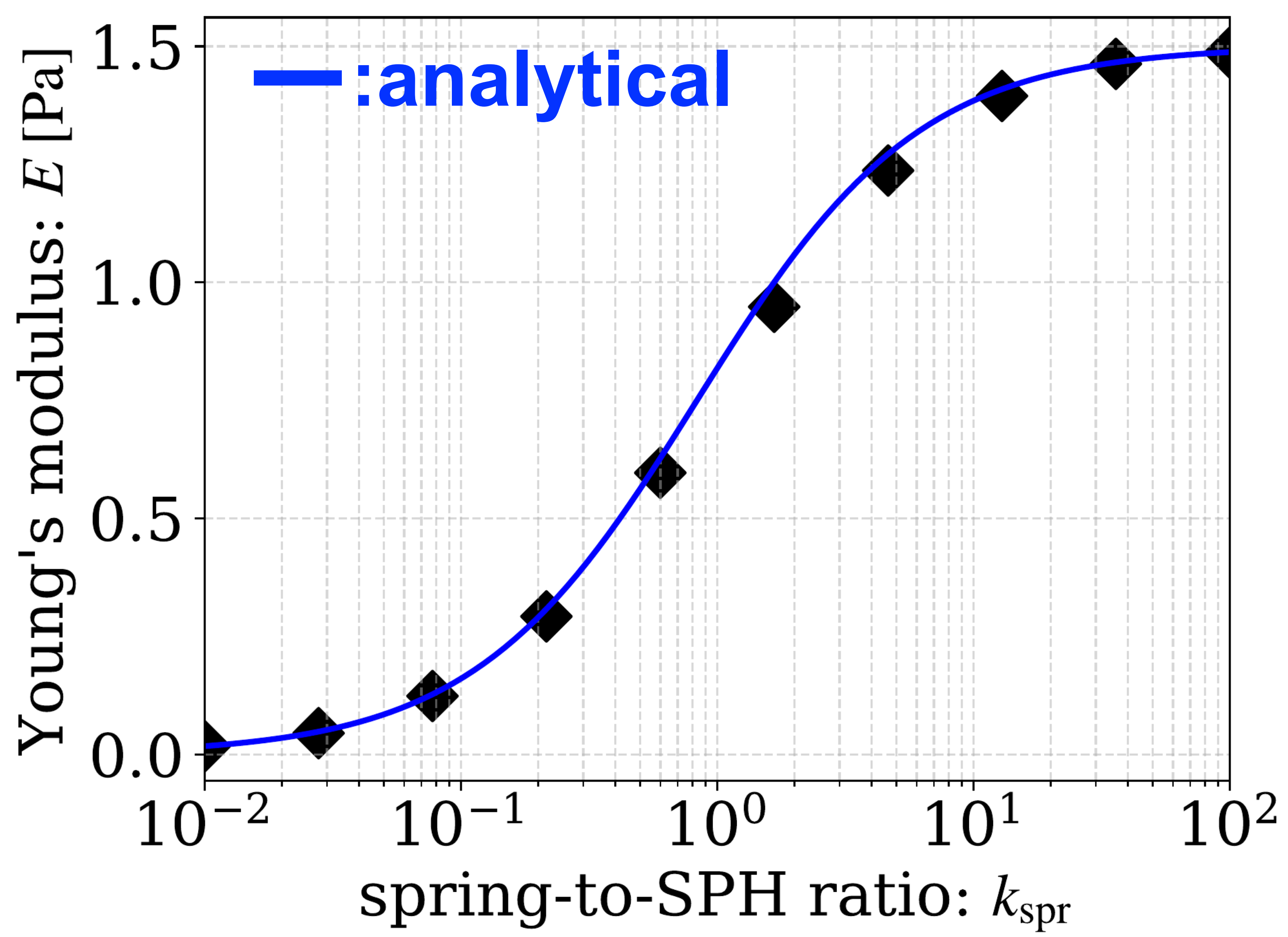
Conservation laws are strictly satisfied, leaving only numerical round-off errors.

Test Calculations

Tensile Testing

σ : Poisson's ratio
E : Young's modulus [Pa]





- Poisson's ratio between 0.25 and 0.5 is achieved

Analytical solution: $\sigma = \frac{1}{4}\frac{k_{\mathrm{spr}} + 5/3}{k_{\mathrm{spr}} + 5/6}, \quad E = \frac{3}{2}\frac{k_{\mathrm{spr}}}{k_{\mathrm{spr}} + 5/6}\mathcal{C}_{\mathrm{s}}^2\rho$

Inverted formla: $k_{\mathrm{spr}} = \frac{5}{6}\frac{1/2 - \sigma}{\sigma - 1/4}, \quad \mathcal{C}_{\mathrm{s}}^2\rho = \frac{2}{3}(1/2 - \sigma)E$

Arbitrary elastic properties can be described

- Poisson's ratio in two directions is the same

Isotropic materials can be described

Our formulation can reproduce **arbitrary linear isotropic elastic materials**

Elastic Linear Waves

σ : Poisson's ratio
E : Young's modulus [Pa]

- Two wave modes in linear perturbation of elastic material
- Measure the sound speed to see if it is consistent with elastic dynamics

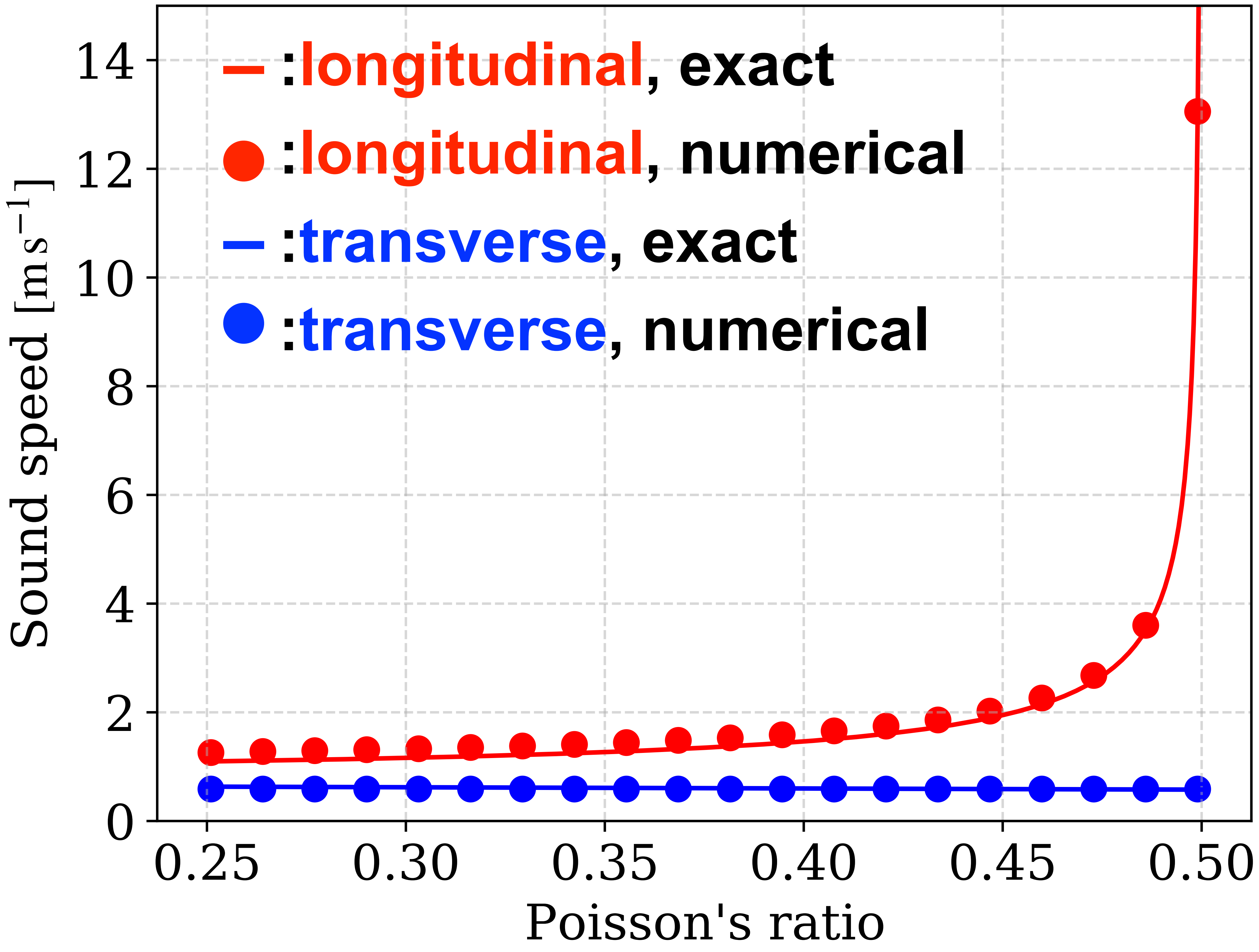
Longitudinal wave: $\boldsymbol{k} \parallel \delta \boldsymbol{A}$

Sound speed: $C_l = \sqrt{\dfrac{E(1-\sigma)}{\rho(1+\sigma)(1-2\sigma)}}$

Transverse wave: $\boldsymbol{k} \perp \delta \boldsymbol{A}$

Sound speed: $C_t = \sqrt{\dfrac{E}{2\rho(1+\sigma)}}$

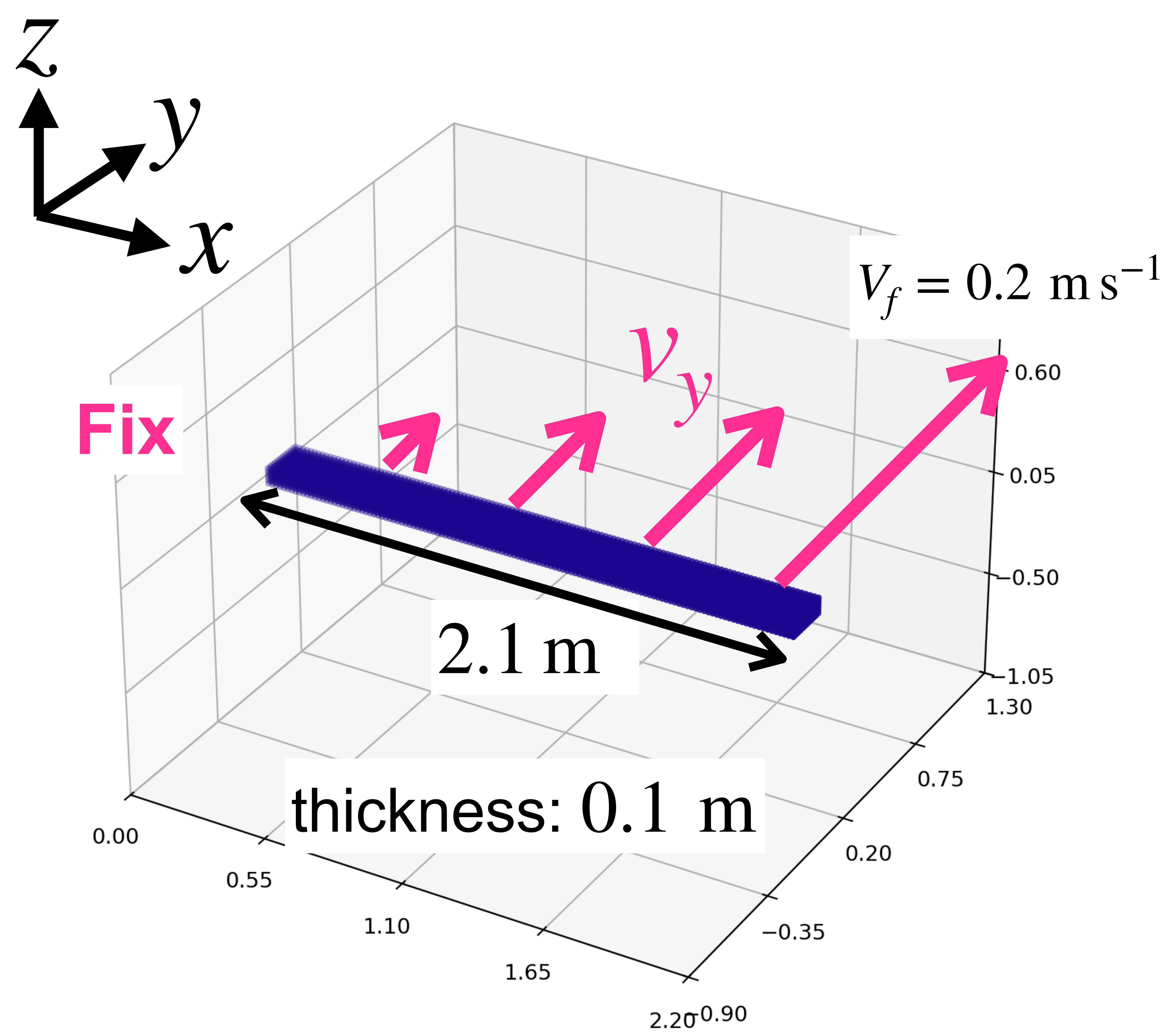
$\boldsymbol{k}$ **:wavenumber** $\delta \boldsymbol{A}$ **:amplitude**



Our formulation can solve linear waves propagation accurately

Oscillating Plate: Setups

- **Vibrate an elastic plate** to see our method's response to finite strain



Number of particles	10^4
Density	$1\ \mathrm{kg\,m^{-3}}$
Side length	$(1,0.1,0.2)\ \mathrm{m}$
Poisson's ratio	0.3
Young's modulus	100 Pa

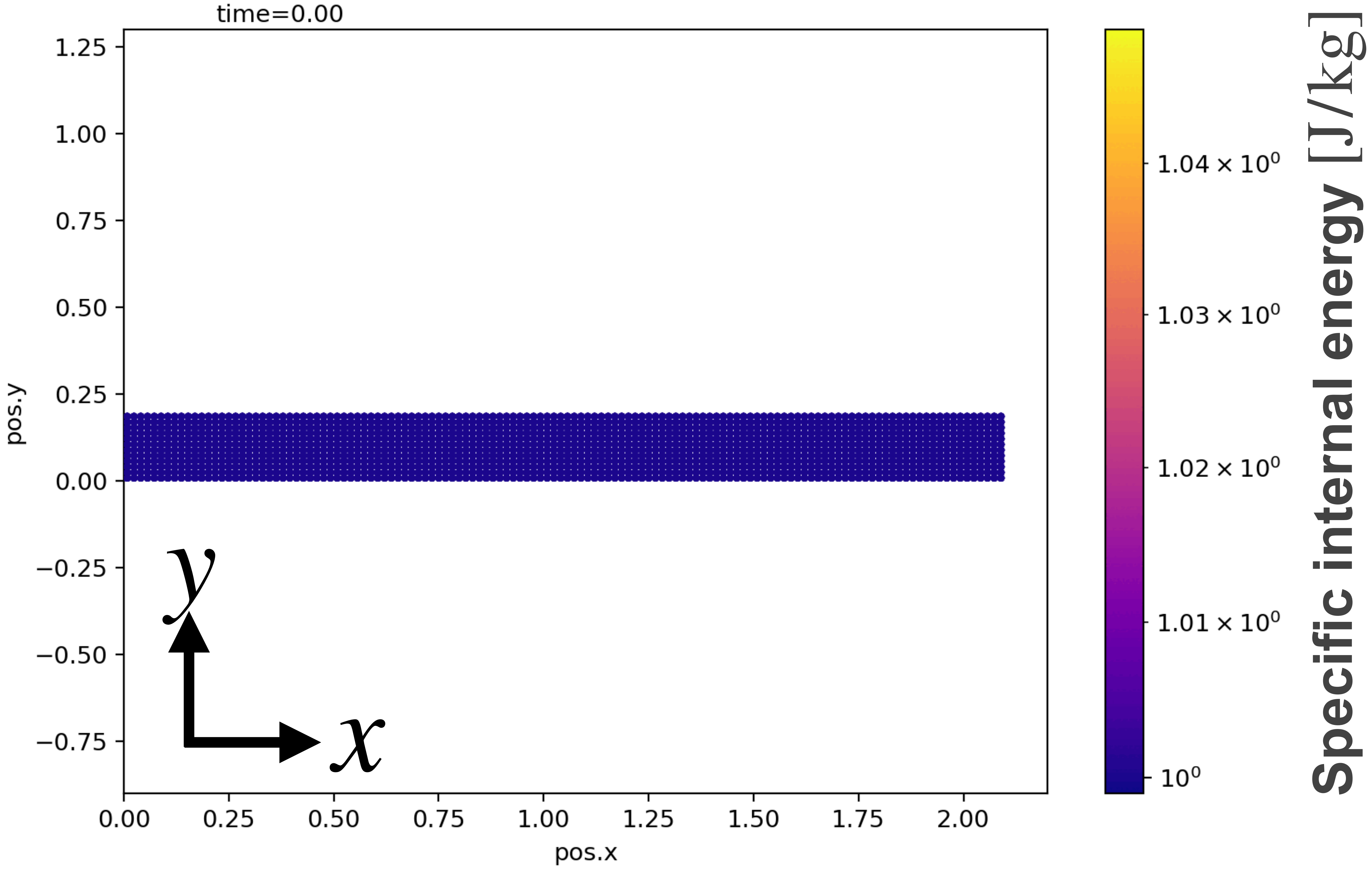
- Initial velocity: [Gray et al., 2001]

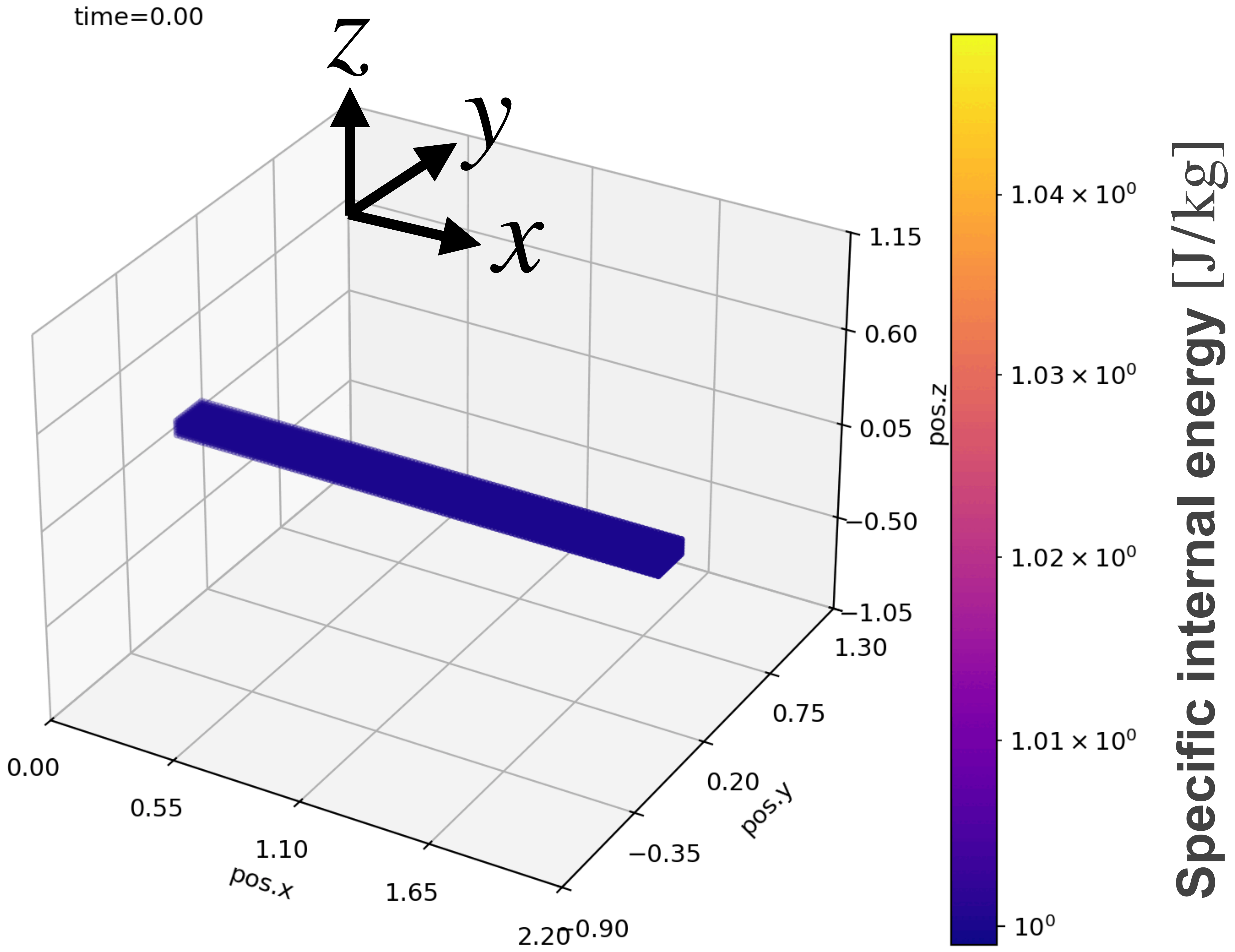
$$\frac{v_y}{C_s} = V_f \frac{[M(\cos(kx) - \cosh(kx)) - N(\sin(kx) - \sinh(kx))]}{Q}$$

$$V_f = 0.1\ [\mathrm{m\,s^{-1}}]$$
$$kL = 1.875$$

$$M = \sin(kL) + \sinh(kL),$$
$$N = \cos(kL) + \cosh(kL),$$
$$Q = 2(\cos(kL)\sinh(kL) - \sin(kL)\cosh(kL))$$

Oscillating Plate: Results





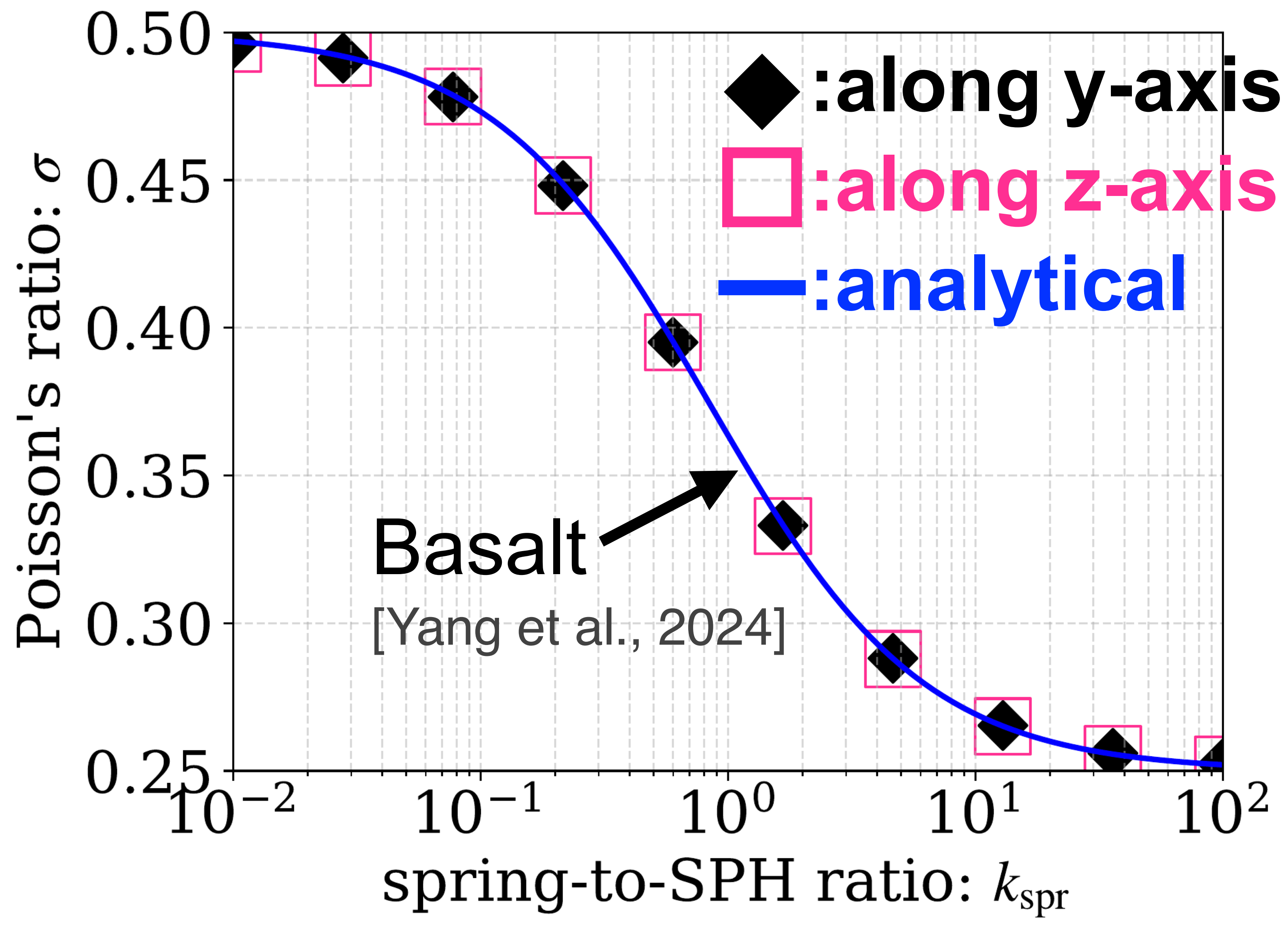
Our formulation can reproduce realistic materials

Summary & Future Prospects

- Smoothed particle hydrodynamics is re-extended to elastic dyanmics

Take-Home Messages:

- Spring-like force is incorporated into SPH
- Significantly reduces computational costs
- Achieves excellent conservation properties



- Future works
  - Incorporating fracture, friction, and porosity effects
  - Describing shock dominated phenomena utilizing Godunov SPH

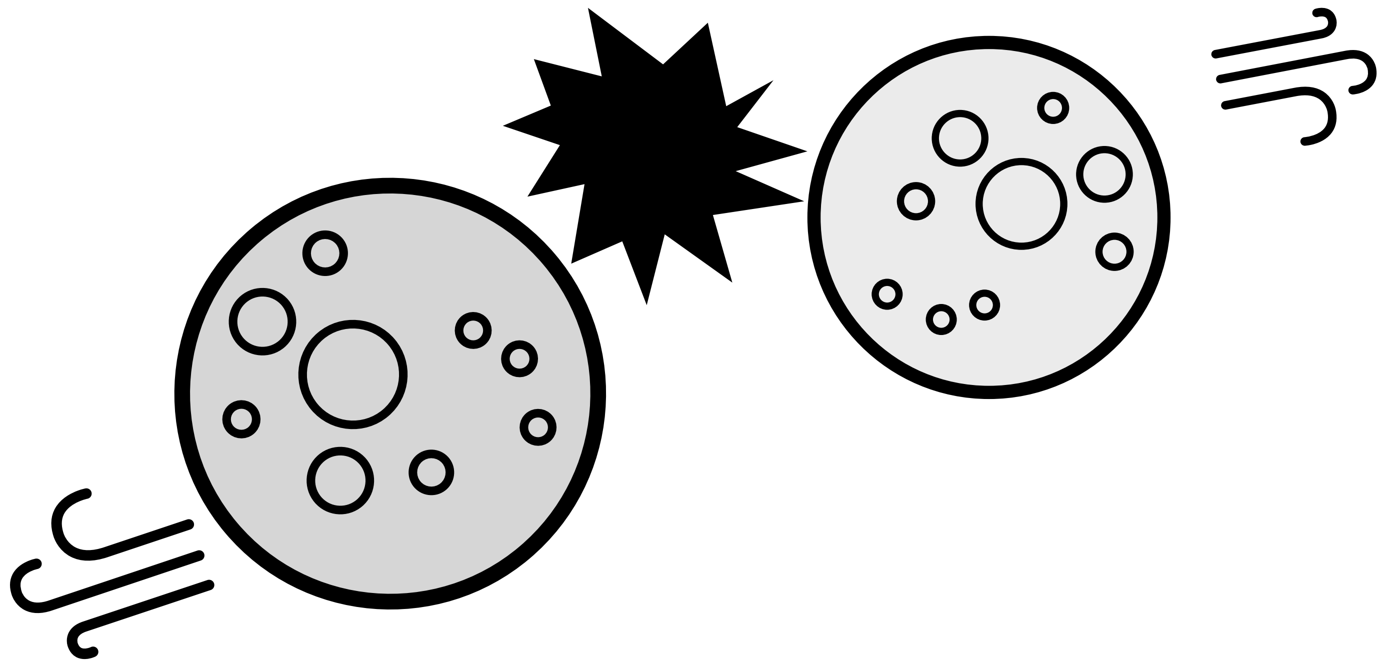
Appendix

Considered Interaction Models

Concept of the model	Poisson's ratio	Linear wave
Asymmetric spring	○	×
Higher-order spring	○	×
Asymmetric + Higher-order spring	○	×
Rest length dependent spring constant	×	○
Stratified spacial distribution of spring	×	○
Density dependent spring constant	×	○
SPH + Spring force	○	○

Forming Solid Planets & Asteroids

Collisions of planetesimals



Merger & Growth

Solid planets e.g., The earth

The earth had globally **melted** in the past

No information about the time of formation is retained

Remnants that did not grow into planets

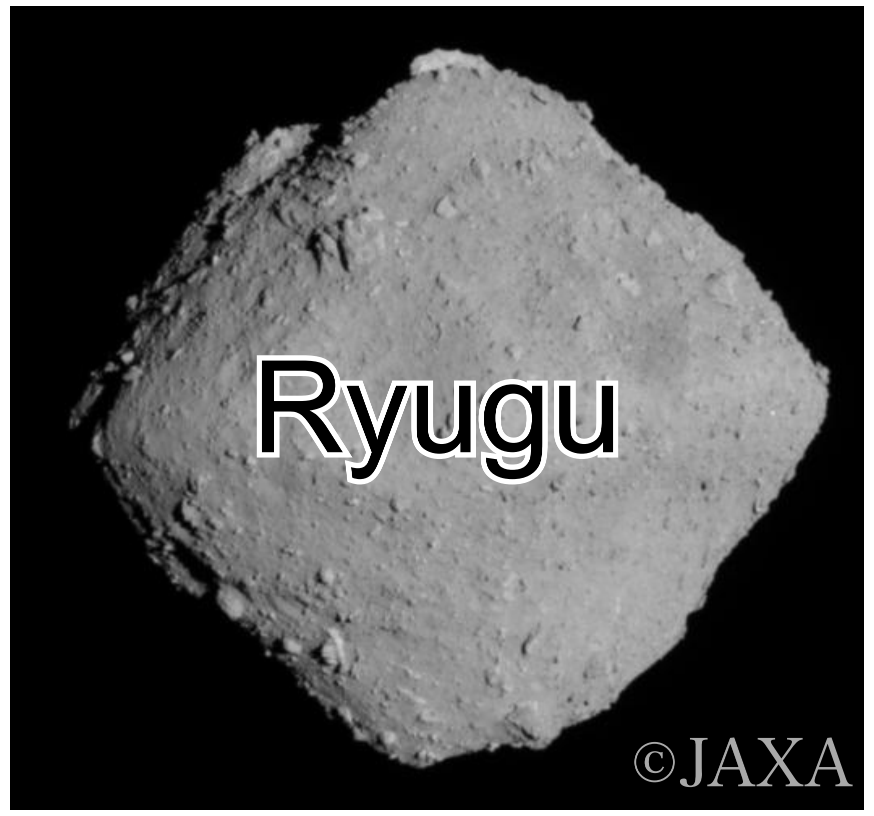
Asteroids are the only entity that **records the history of planetary formation**

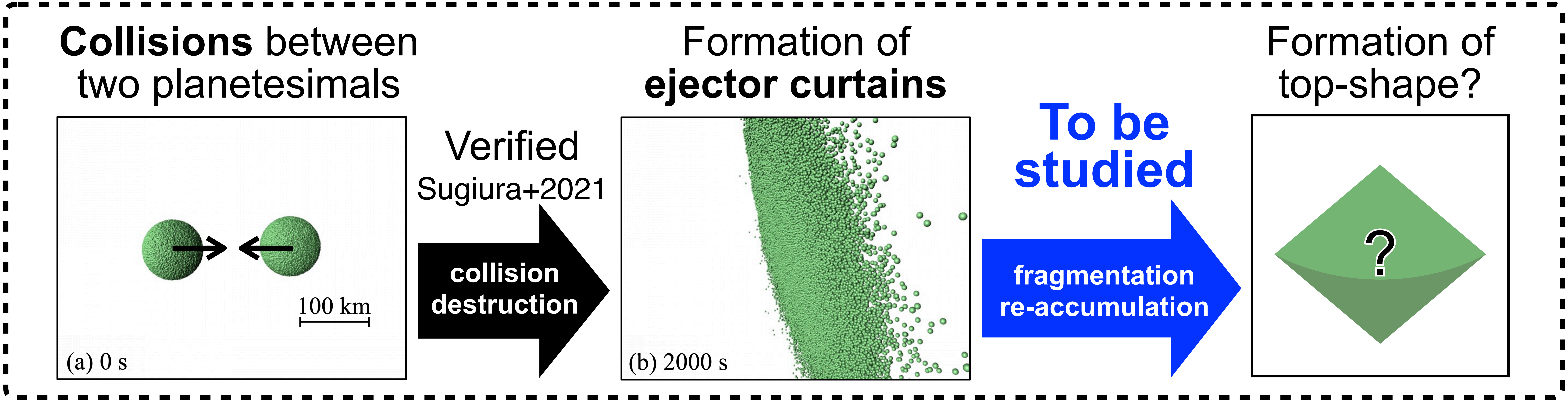
Studying the **formation of asteroids** through collision phenomena would reveal…

- The process of forming **solid planets**
- The evolution of **planetary systems**

Formation of Top-shaped Asteroids

- The formation mechanism for **top-shaped asteroids** is not revealed sufficiently.
- A promising scenario dominated by **angular momentum** is suggested, but has not yet been verified.

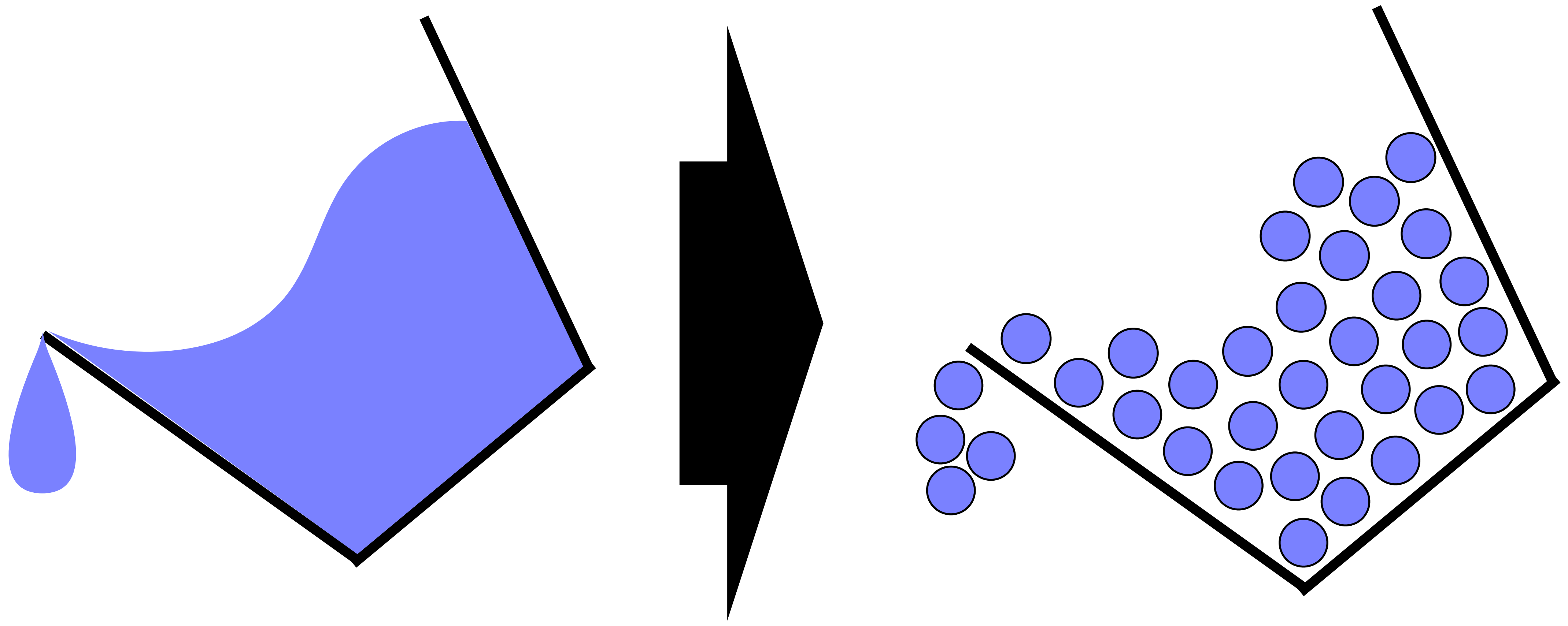




Simulations with methods that strictly conserve angular momentum are required.

Smoothed Particle Hydrodynamics

- **Smoothed Particle Hydrodynamics (SPH)** is a numerical method to describe fluids.
- Continuous fields of physical quantities are discretized in space by **super particles**.



A real fluid

Particle system

Particle's quantities

- **Density** (for standard SPH)

$$\rho_i = \sum_j m_j W(|\boldsymbol{x}_i - \boldsymbol{x}_j|, h)$$

Particle's mass: m_j

Smoothing length: h

- **Other quantities**

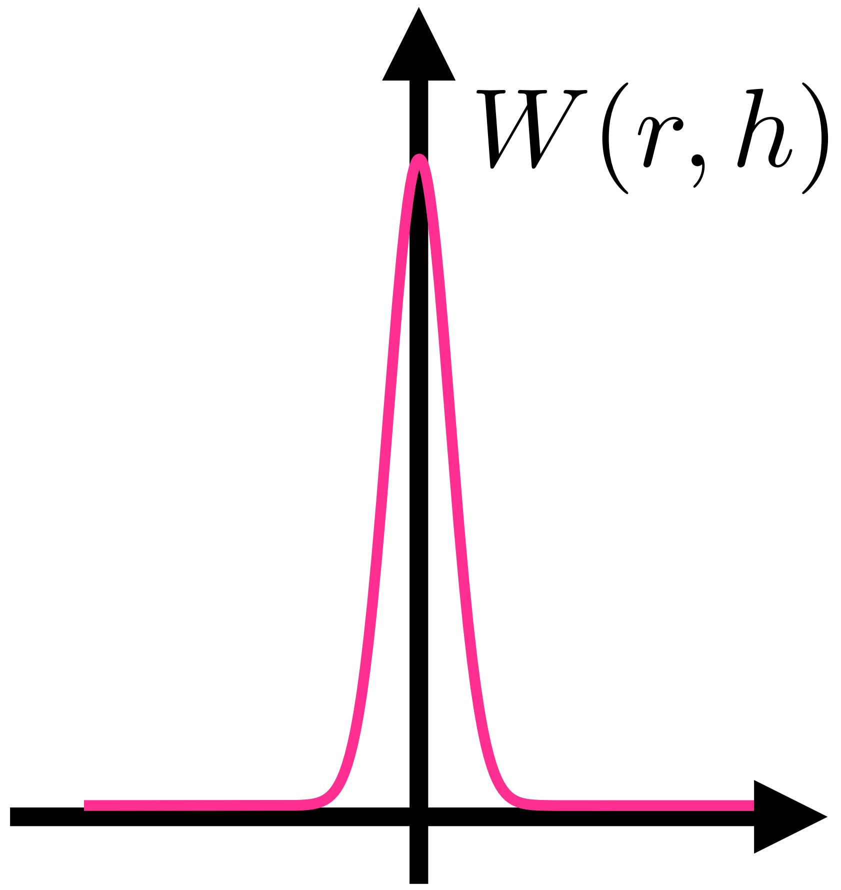
$$f_i = \int dV\, f(\boldsymbol{x}) W(|\boldsymbol{x} - \boldsymbol{x}_i|, h)$$

- **Kernel function**

$$\lim_{r\to\infty} W(r,h) = 0$$

$$W(-r,h) = -W(r,h)$$

$$\int_0^\infty dr\, 4\pi r^2\, W(r,h) = 1$$



Basic Equations for SPH

Particle's quantities

■ Density **(for standard SPH)**

$$\rho_i = \sum_j m_j W(|\boldsymbol{x}_i - \boldsymbol{x}_j|, h)$$

■ Other quantities

$$f_i = \int dV\, f(\boldsymbol{x}) W(|\boldsymbol{x} - \boldsymbol{x}_i|, h)$$

■ Kernel function

$$\lim_{r\to\infty} W(r,h) = 0$$

$$W(-r,h) = -W(r,h)$$

$$\int_0^\infty dr\, 4\pi r^2\, W(r,h) = 1$$

Equations for continuous field

Continuity:

$$\frac{d\rho}{dt} = -\rho \frac{\partial}{\partial x^\alpha} v^\alpha$$

Motion:

$$\frac{dv^\alpha}{dt} = -\frac{1}{\rho}\frac{\partial P}{\partial x^\alpha}$$

Energy:

$$\frac{du}{dt} = -\frac{P}{\rho}\frac{\partial}{\partial x^\alpha} v^\alpha$$

Discretize in space

Equations for SPH

$$m_i \frac{d\dot{\boldsymbol{x}}_i}{dt} = -\sum_j \left[P {V_{\mathrm{p}}}^2\right]_{ij} \frac{\partial}{\partial \boldsymbol{x}_i} W(|\boldsymbol{x}_i - \boldsymbol{x}_j|, h) \quad V_{\mathrm{p},i} = \frac{m_i}{\rho_i}$$

$$m_i \frac{du_i}{dt} = \frac{1}{2}\sum_j \left[P {V_{\mathrm{p}}}^2\right]_{ij} (\dot{\boldsymbol{x}}_i - \dot{\boldsymbol{x}}_j) \cdot \frac{\partial}{\partial \boldsymbol{x}_i} W(|\boldsymbol{x}_i - \boldsymbol{x}_j|, h)$$

For standard SPH: $\left[P {V_{\mathrm{p}}}^2\right]_{ij} = m_i m_j \left(\frac{P_i}{{\rho_i}^2} + \frac{P_j}{{\rho_j}^2}\right)$

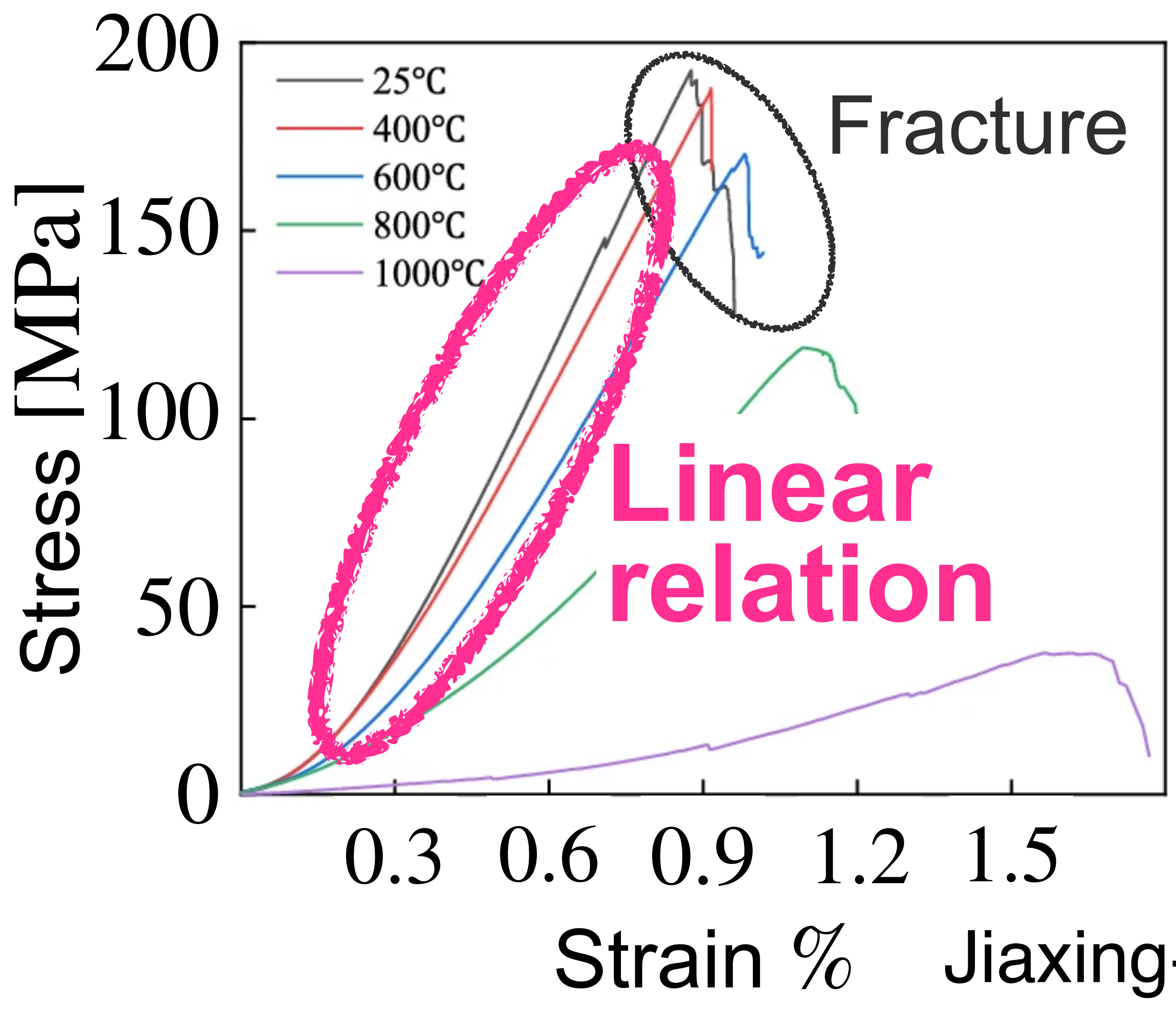
Linear Isotropic Elasticity

- We assume **linear isotropic** materials.

☐ **Linear** relation between **stress** & **strain**
☐ **Isotropic** properties of material

- Arbitrary linear isotropic elastic bodies are characterized by two parameters: **Poisson's ratio & Young's modulus**.

Stress-Strain Curve for Basalt



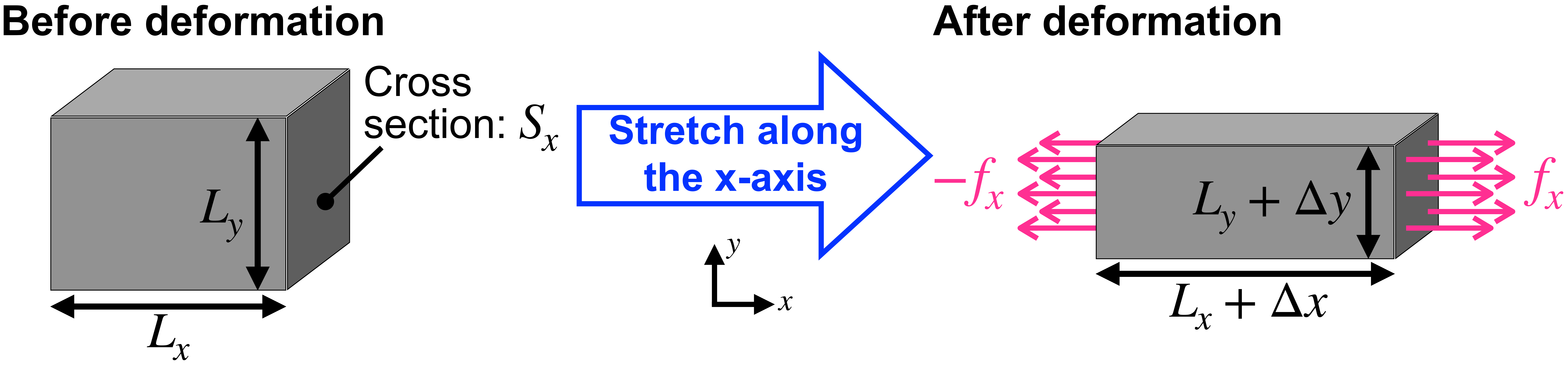
Jiaxing+ 2023

Existing central force models cannot accomplish Poisson's ratio grater than 1/4. e.g. Donze+ 1995; Nayfeh+ 1978

We shall make sure the various **Poisson's ratios** of our elasicity model.

Tensile Test & Elastic Parameters

- Tensile test is to measure **Poisson's ratio & Young's modulus**.



Elastic parameters for linear isotropic material

Poisson's ratio:

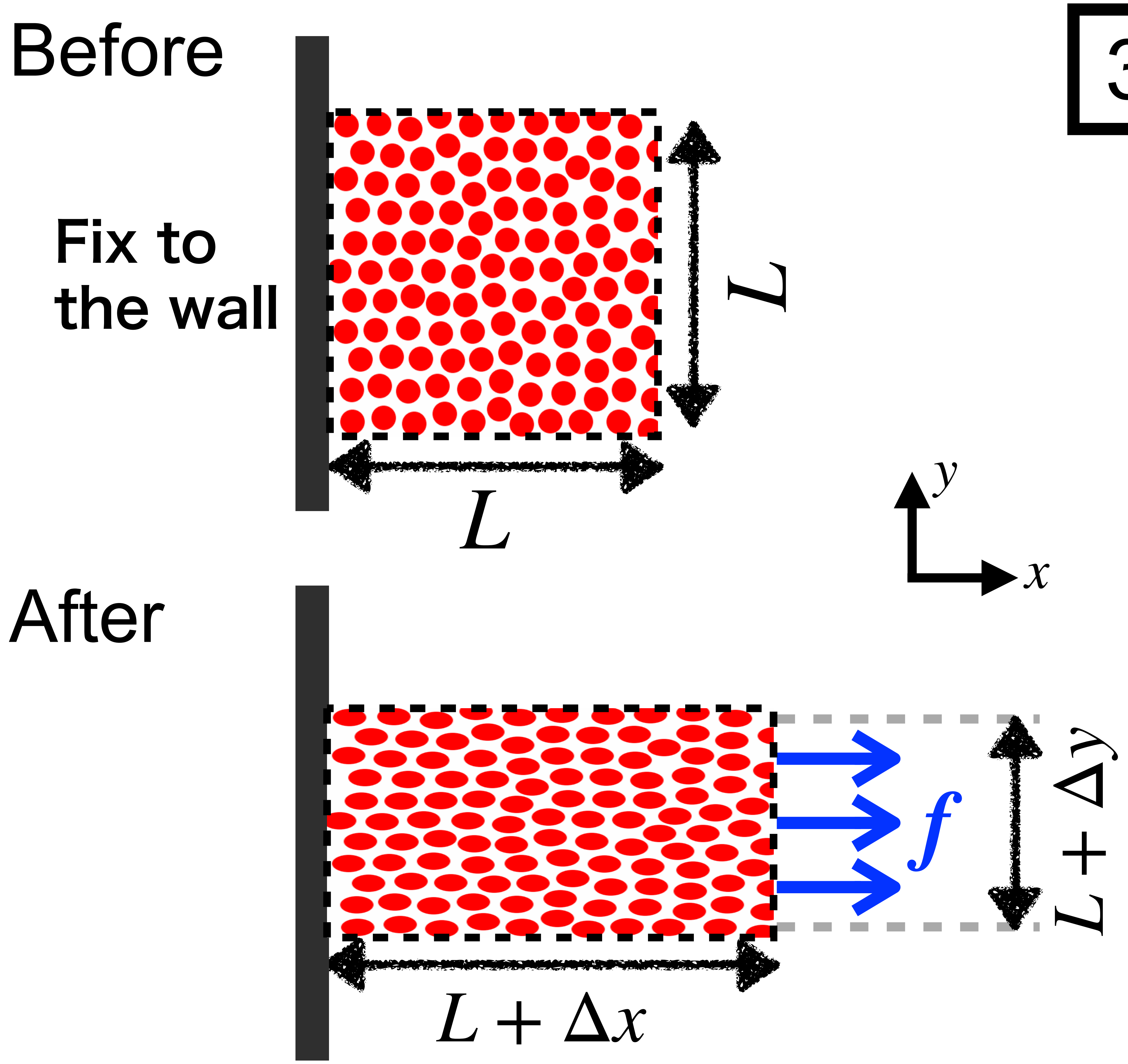
$$\frac{\text{(Strain perpendicular to tensile axis)}}{\text{(Strain parallel to tensile axis)}} = -\frac{\Delta y / L_y}{\Delta x / L_x}$$

Young's modulus:

$$\frac{\text{(Stress parallel to tensile axis)}}{\text{(Strain parallel to tensile axis)}} = -\frac{f_x / S_x}{\Delta x / L_x} \ \text{[Pa]}$$

Tensile Test: Setups

- Stretch material along x-axis, measure Δx, Δy, and derive Poisson's ratio & Young's modulus



3-D calculation

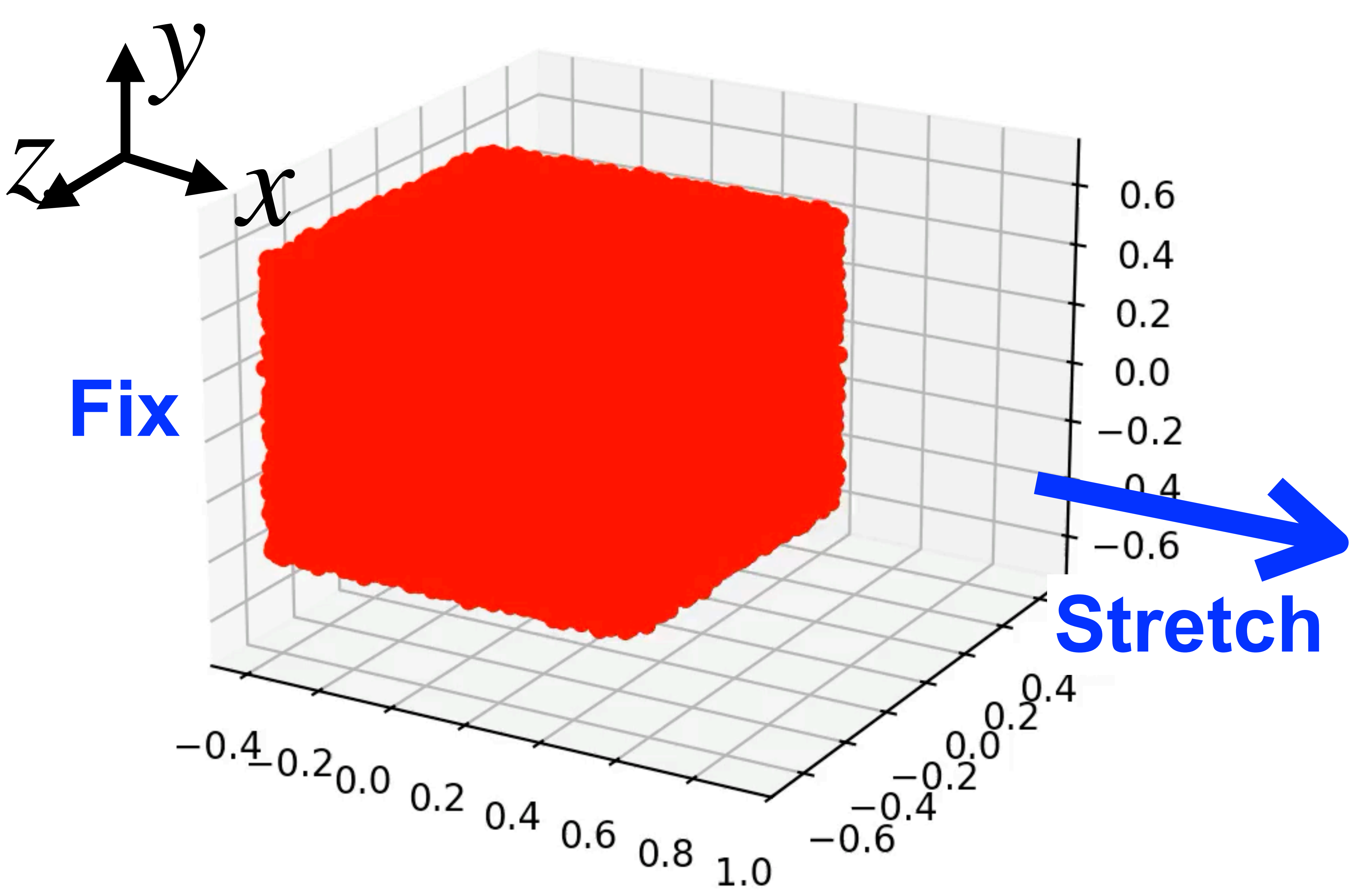
Number of particles	10^5
Density	$1\ \mathrm{kg\,m^{-3}}$
Side length	$1\ \mathrm{m}$
Stretch length	$0.01\ \mathrm{m}$
Spring-to-SPH ratio	$10^{-2} - 10^{2}$
SPH bulk sound speed	$1\ \mathrm{m\,s^{-1}}$

Parallelization: FDPS
Iwasawa+ 2016, Namekata+ 2018

Tensile Test: Setups

- Stretch material along x-axis, measure Δx, Δy, and derive Poisson's ratio & Young's modulus

Example of exaggerated tensile



Number of particles	10^5
Density	$1\ \mathrm{kg\,m^{-3}}$
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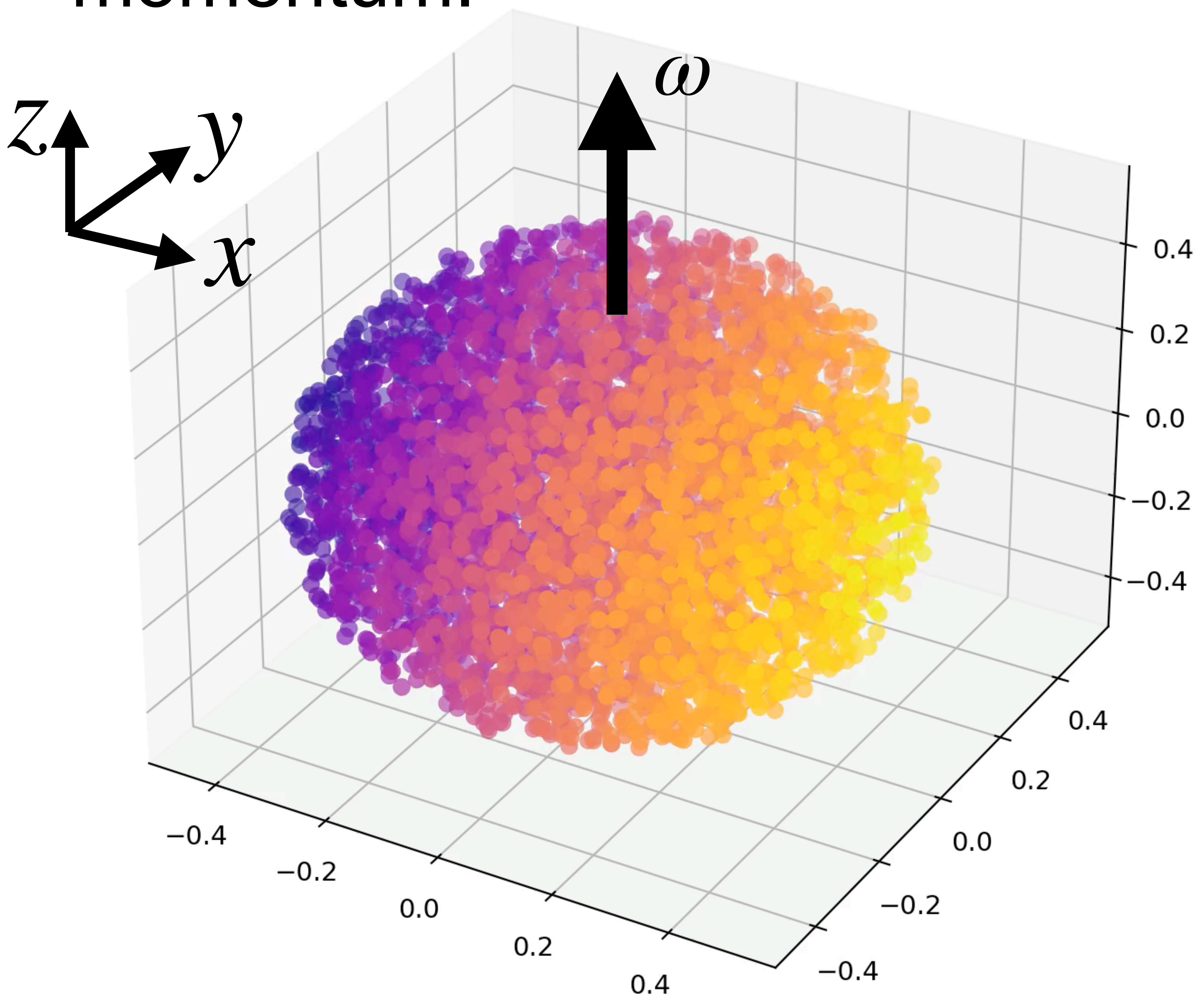
Stretch length	$0.01\ \mathrm{m}$
Spring-to-SPH ratio	$10^{-2} - 10^2$
SPH bulk sound speed	$1\ \mathrm{m\,s^{-1}}$

Parallelization: FDPS

Iwasawa+ 2016, Namekata+ 2018

Rotating Sphere: Setups

- **Rotate a sphere material** to see how our method conserves angular momentum.



- Initial conditions for **aluminium**

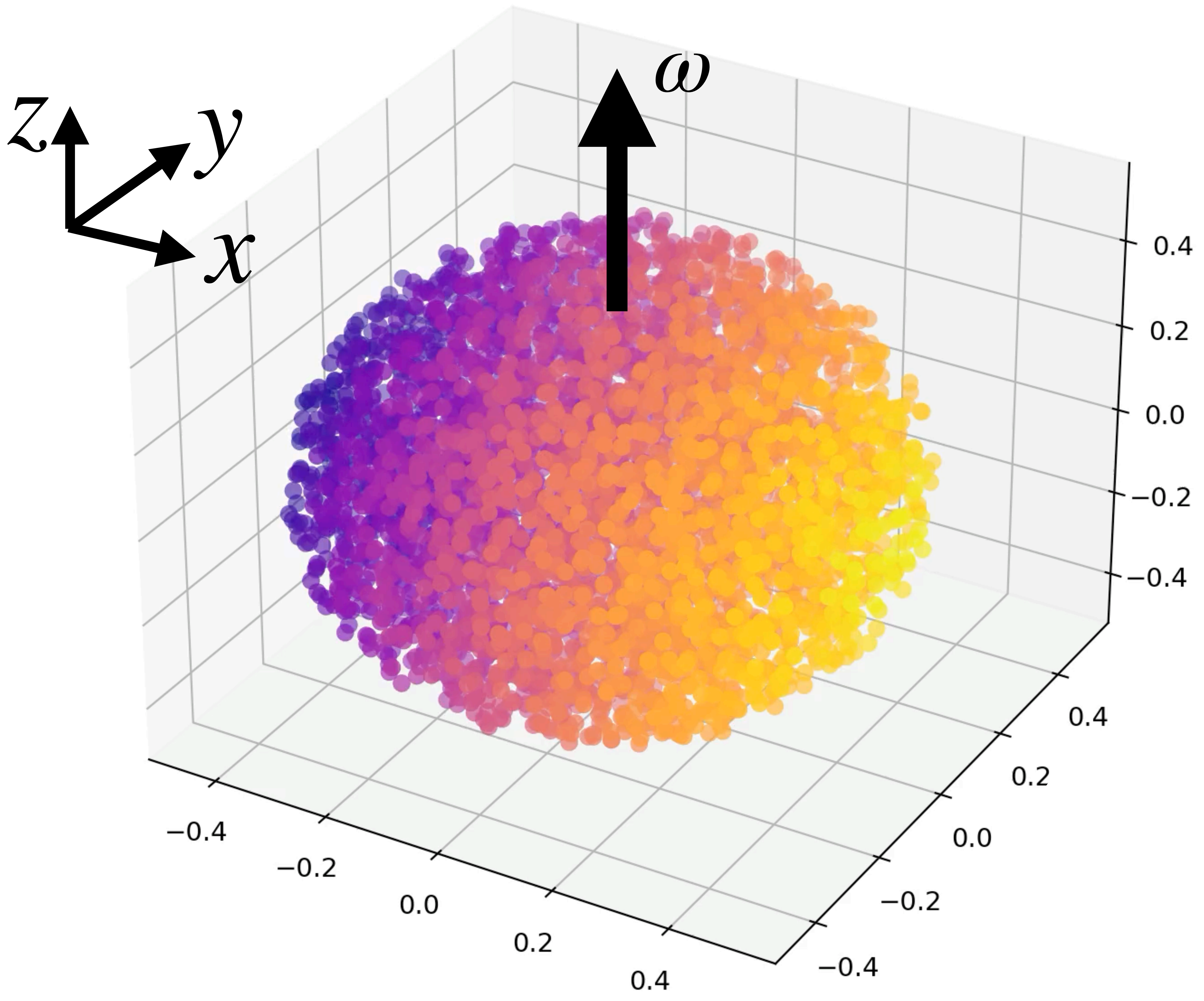
Number of particles	10^4
Density	$27\ \mathrm{kg\,m^{-3}}$
Side length	$0.5\ \mathrm{m}$
Bulk modulus	$75.5\,\mathrm{GPa}$
Shear modulus	$26.1\,\mathrm{GPa}$
Angular velocity	$10^4\,\mathrm{s^{-1}}$

Set initial velocities for rigid body rotation

$$\boldsymbol{v} = \boldsymbol{r} \times \boldsymbol{\omega}$$

Rotating Sphere: Results

Color: initial x position



Final angular momentum error:

$$\left| \frac{L_{x,\mathrm{init}} - L_{x,\mathrm{final}}}{L_{x,\mathrm{init}}} \right| \sim 0.0001\ \%$$

Our method is indeed capable of conserving angular momentum.

Average computational time per step:

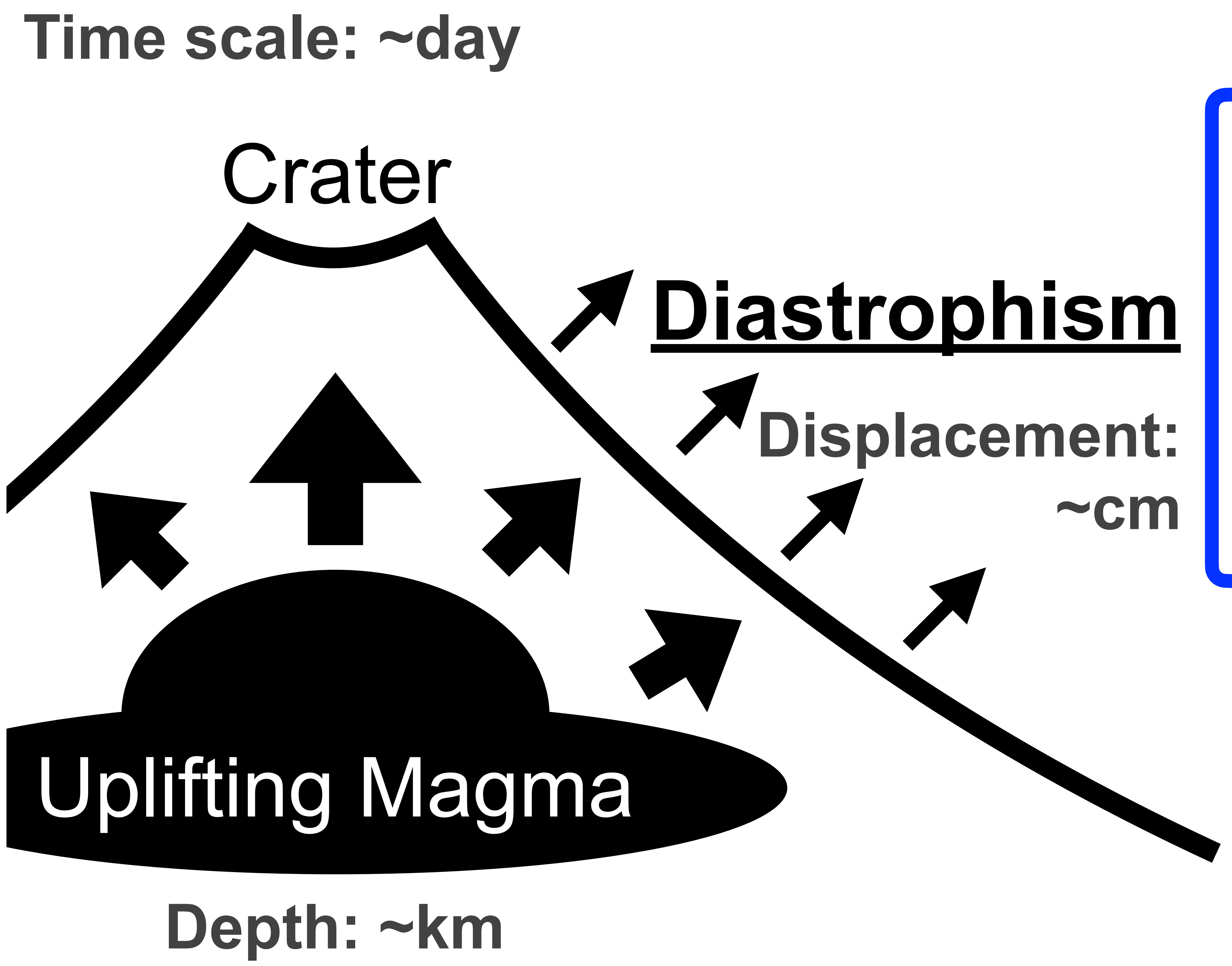
2.37e-02 sec

7.6 times faster!

For existing method: **1.80e-01 sec**

Volcanic Eruption & Diastrophism

- Volcanic eruptions are caused by **uplifting magma** by buoyancy.
- **Rising magma causes diastrophism** by pushing the ground.



A Method to Predict Eruptions

① Measure the **distribution of diastrophism**

② Forecast the distribution of underground magma

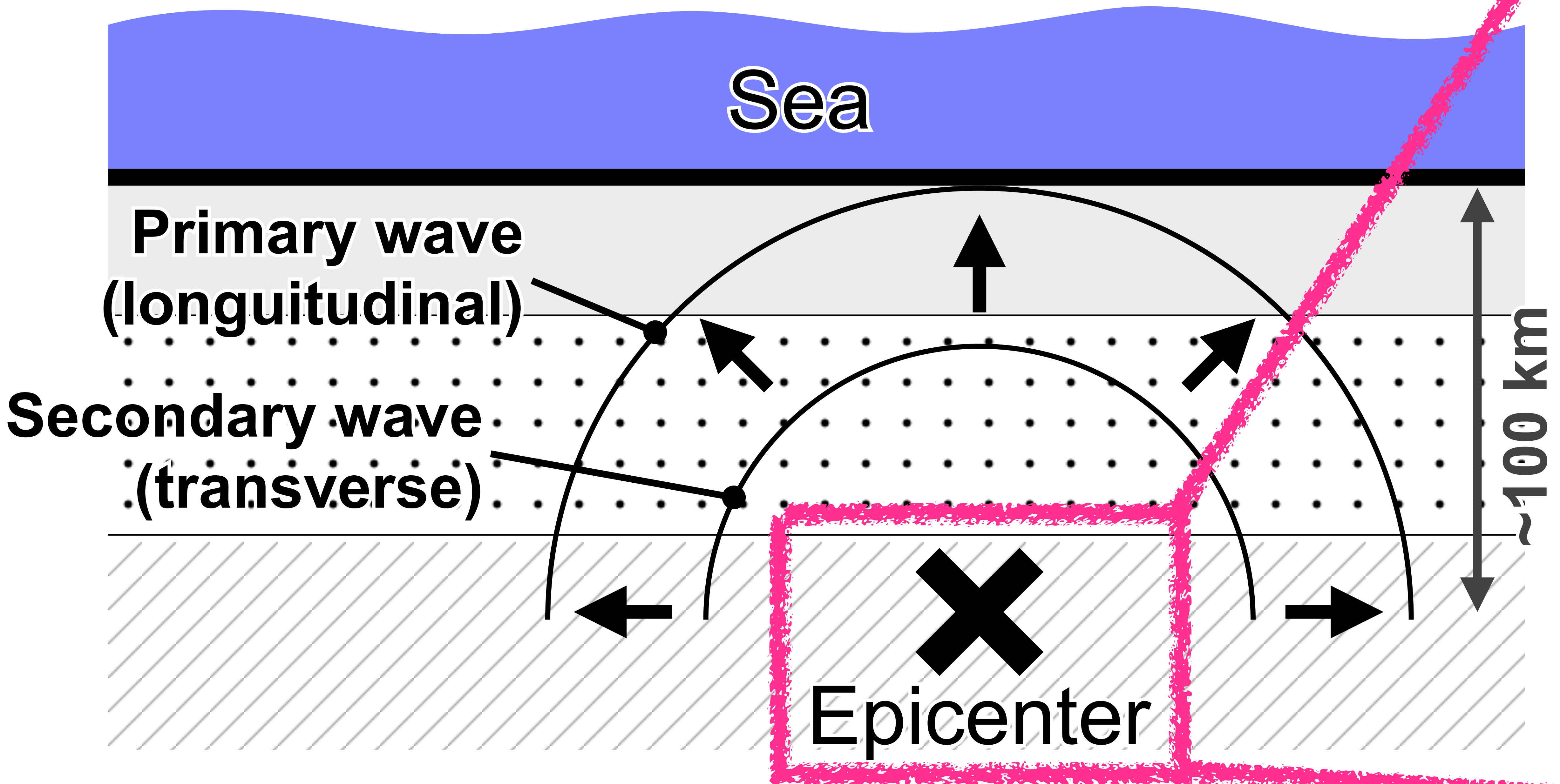
③ Estimate the **time to erupt and the magnitude**

A fast numerical method to solve the behavior of **two objects, magma and rock** is required.

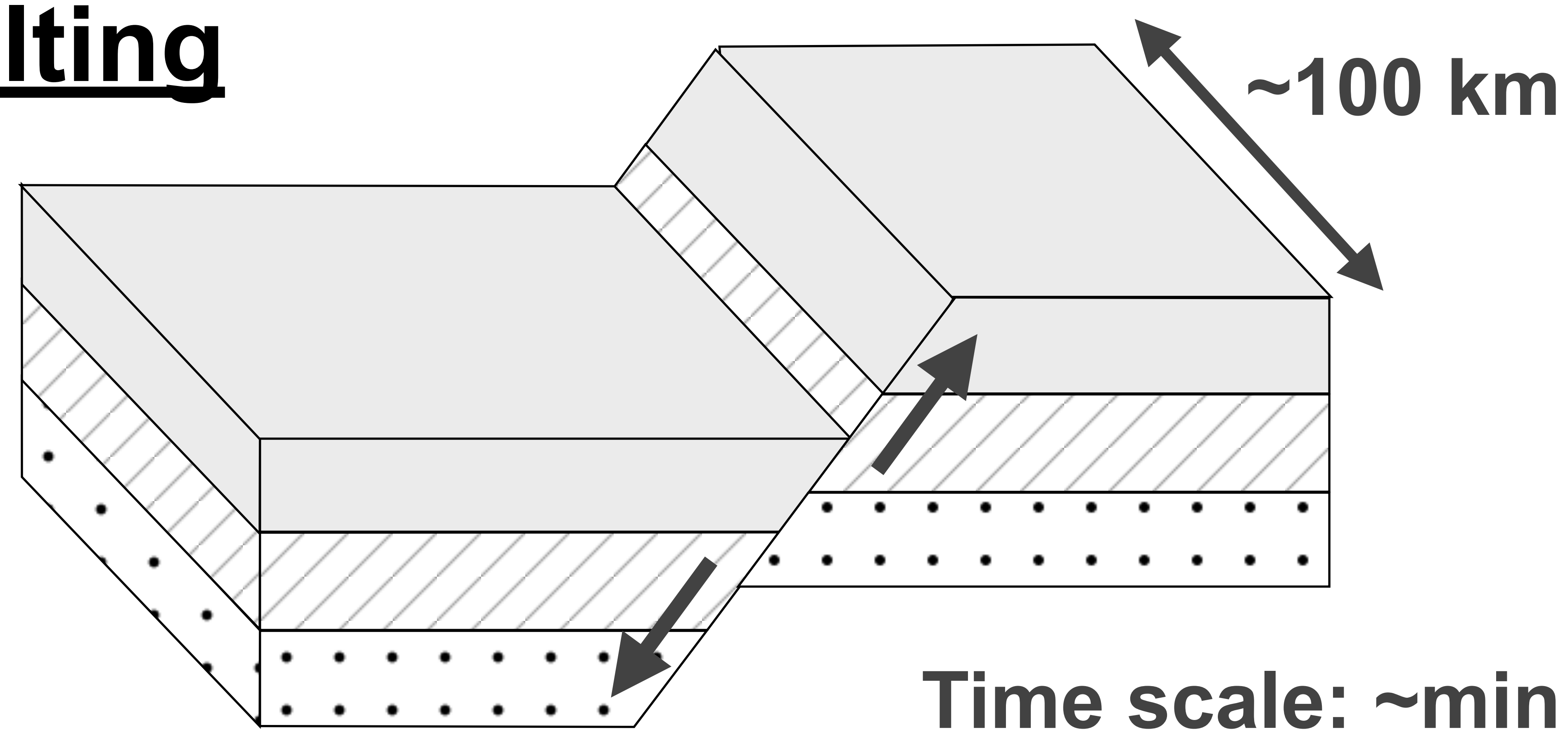
Numerical calculation with our method would predict volcanic eruptions.

Seismic Activity Caused by Faulting

- Seismic activity in general is caused by **faulting in the subsurface**.
- However, the most previous studies **assumed a point epicenter**.



Faulting



Simulating faulting with existing methods becomes challenging due to the **significant deformation**, which leads to **insufficient understanding**.

Our method enables consistent simulations without assuming a point epicenter.

Practical Solutions to Real Problems

- With low resolution axis, the accuracy of the Poisson ratio deteriorates

- Pouring particles into geometrically narrow/thin regions
- Using particles of non-uniform mass

Model Parameters & Poisson's Ratio

Poisson比

$$\sigma = \frac{1}{2}\left(1 - \frac{1}{1/3 + K/\mu}\right)$$

$K \to \infty$ で , 最大値 $1/2$ になる

Poisson比 $1/2$

→ 微小変形で体積一定

→ 非圧縮 (流体と同じ挙動)

応力テンソル

$$\sigma_{ik} = K\delta_{ik}u_{ll} + 2\mu\left(u_{ik} - \frac{1}{3}\delta_{ik}u_{ll}\right)$$

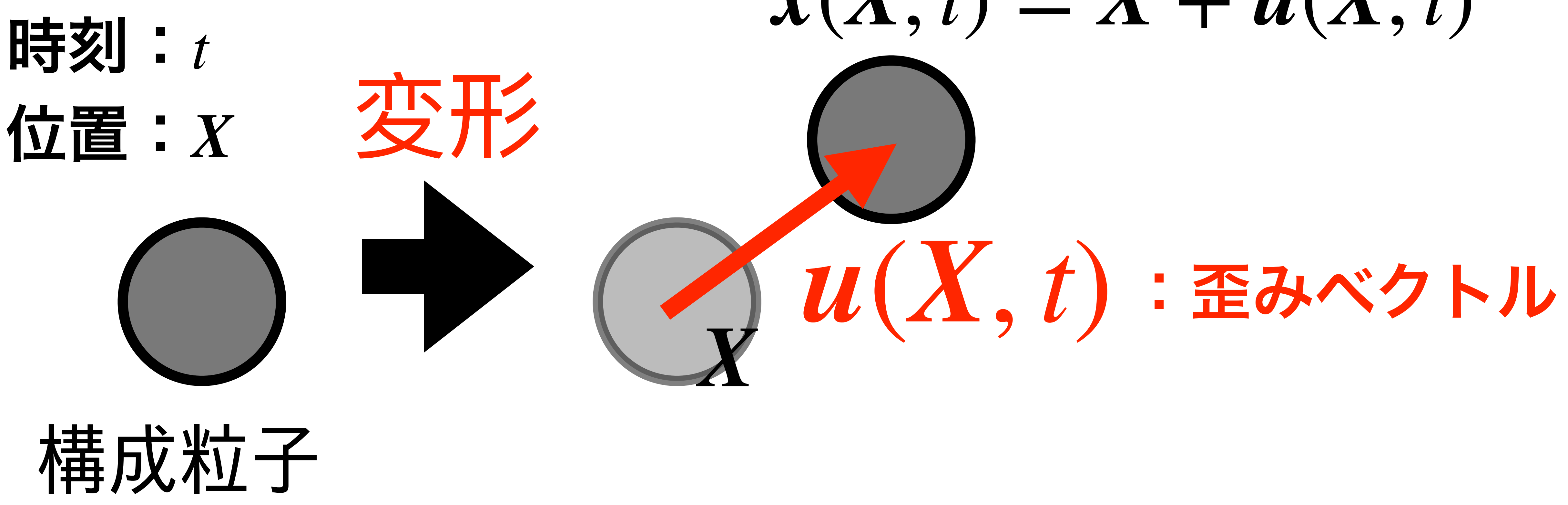
K : 体積弾性率 , μ : 剪断弾性率

Elastic Dynamics: Strain Tensor

- 弾性体力学： 物質の変形を記述

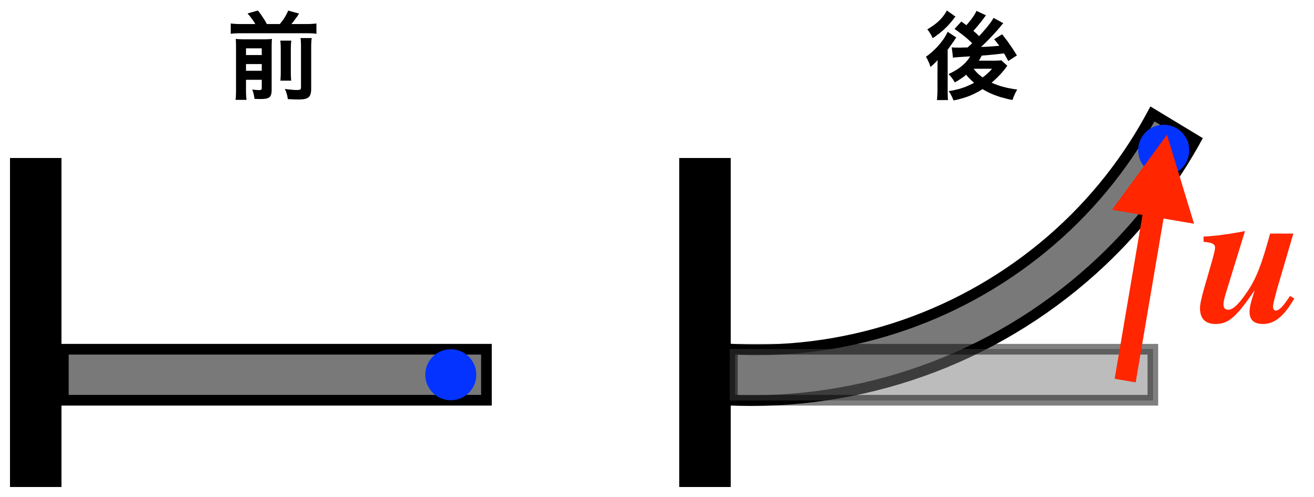
歪みベクトル：変形による構成粒子の変位

$$\boldsymbol{x}(\boldsymbol{X}, t) = \boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X}, t)$$



全体を見た場合：

E.g. 棒の曲げ



歪みテンソル：

歪みの変化率から剛体回転を抜いたもの

$$\epsilon^{\alpha\beta} = \frac{1}{2}\left(\frac{\partial u^\alpha}{\partial x^\beta} + \frac{\partial u^\beta}{\partial x^\alpha}\right)$$

剛体回転の寄与は自動で消える

歪み速度テンソル：

$$\dot{\epsilon}^{\alpha\beta} = \frac{1}{2}\left(\frac{\partial v^\alpha}{\partial x^\beta} + \frac{\partial v^\beta}{\partial x^\alpha}\right)$$

Elastic Dynamics: Constitutive Eq.

- 構成則：物質固有の力学特性を表現

$$\underset{\text{応力}}{\sigma^{\alpha\beta}} = \underset{\text{歪み}}{C^{\alpha\beta}(\epsilon)}$$

$C^{\alpha\beta}$：歪みを引数にもつ関数

$$\epsilon^{\alpha\beta} = \frac{1}{2}\left(\frac{\partial u^\alpha}{\partial x^\beta} + \frac{\partial u^\beta}{\partial x^\alpha}\right)$$

一般の弾性体

任意関数：$C^{\alpha\beta}$

$$\sigma^{\alpha\beta} = C^{\alpha\beta}(\epsilon)$$

自由度：無限

線形弾性体

Hookの法則

$$\sigma^{\alpha\beta} = C^{\alpha\beta\gamma\delta}\epsilon^{\gamma\delta}$$

自由度：21

２個の弾性パラメータ：

- Poisson比
- Young率

自由度：2

等方弾性体

等方的な関係式

特徴的方向なし

自由度：無限

線形等方弾性体の再現には，２個のパラメータを持つ弾性体モデルが必要

Poisson比＆Young率

Elastic Dynamics: Covariance

- 客観性の原理：物質の特徴は，観測する系に依らない

  →**剛体並進**や**剛体回転**に対して，物質静止系から見れば不変である．

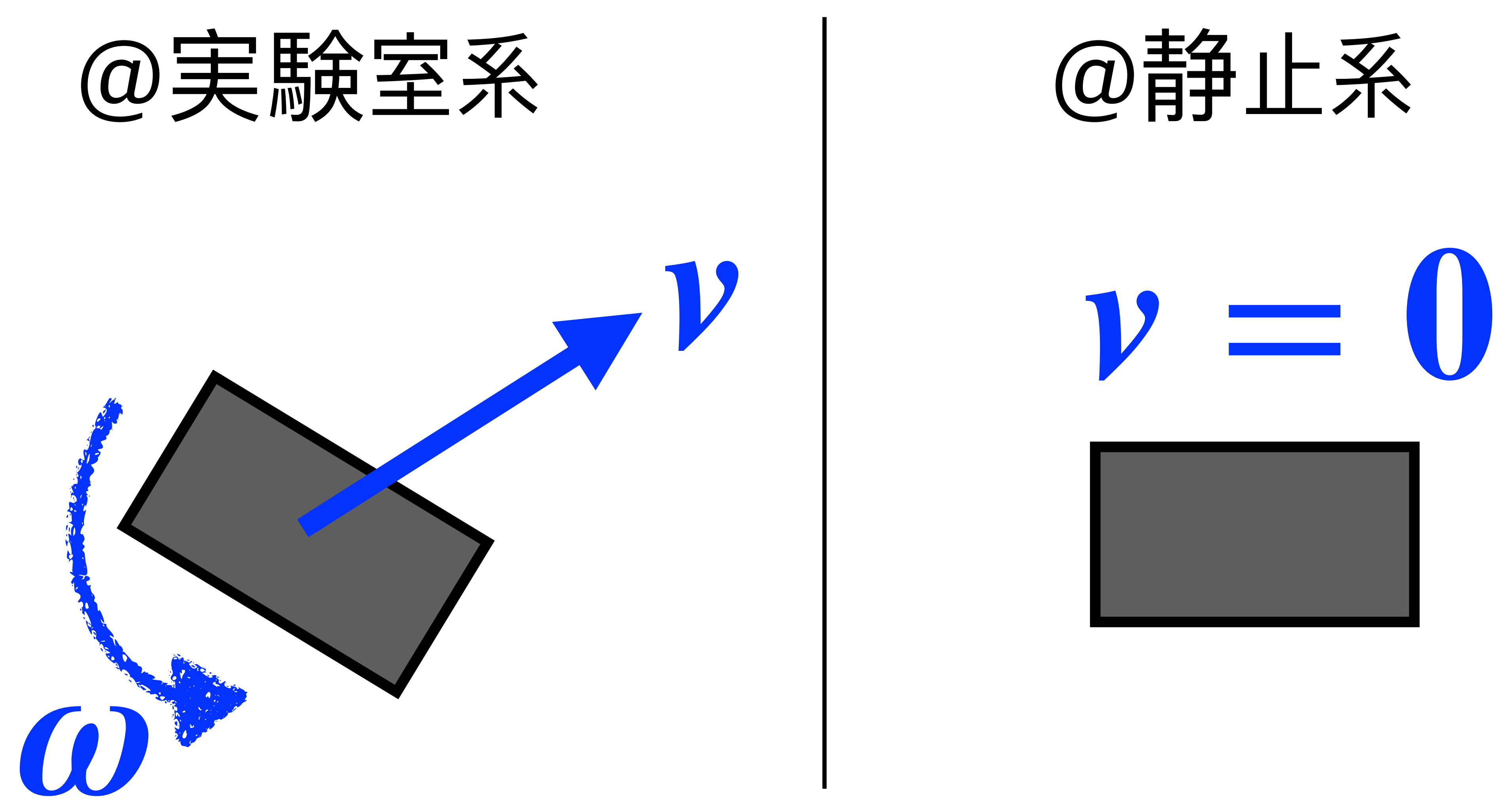
座標変換：$\boldsymbol{x}' = \boldsymbol{c}(t) + Q(t)\boldsymbol{x}$　　**直交行列**：$Q^{-1} = Q^T$

- 客観性を満たすように再定義

  Jaumannの応力速度

$$\dot{\sigma}_{[J]} = \dot{\sigma} - R\sigma + \sigma R$$

$$R^{\alpha\beta} = \frac{1}{2}\left(\frac{\partial}{\partial x^\beta}v^\alpha - \frac{\partial}{\partial x^\alpha}v^\beta\right)$$



硬い箱が変形なしで飛んでいる → **静止系で見ても当然変形なし**