Relativistic Jet Propagation into Vacuum Simulated with the Godunov-Type SPH Method

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Nagoya Univ. (Japan)
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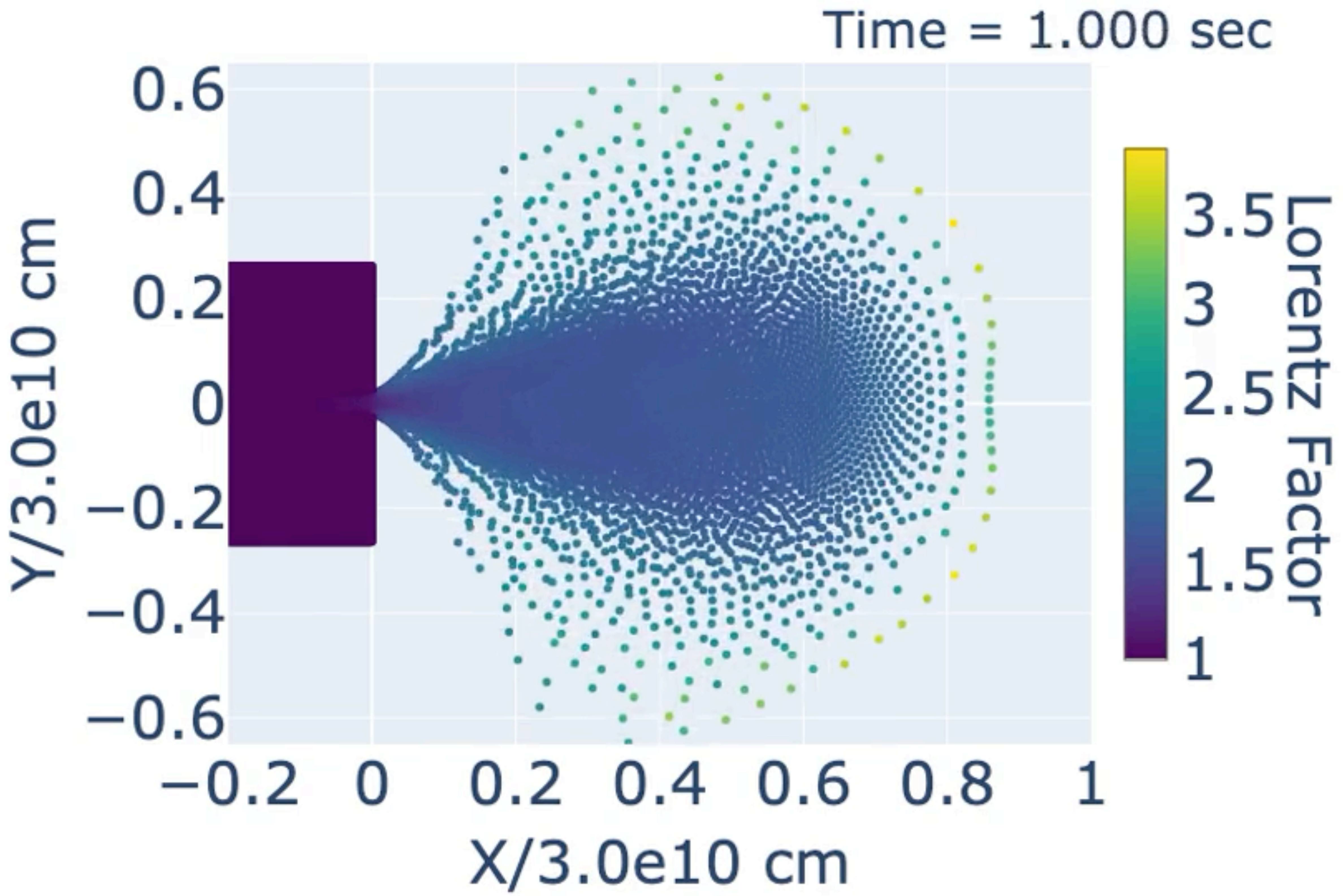
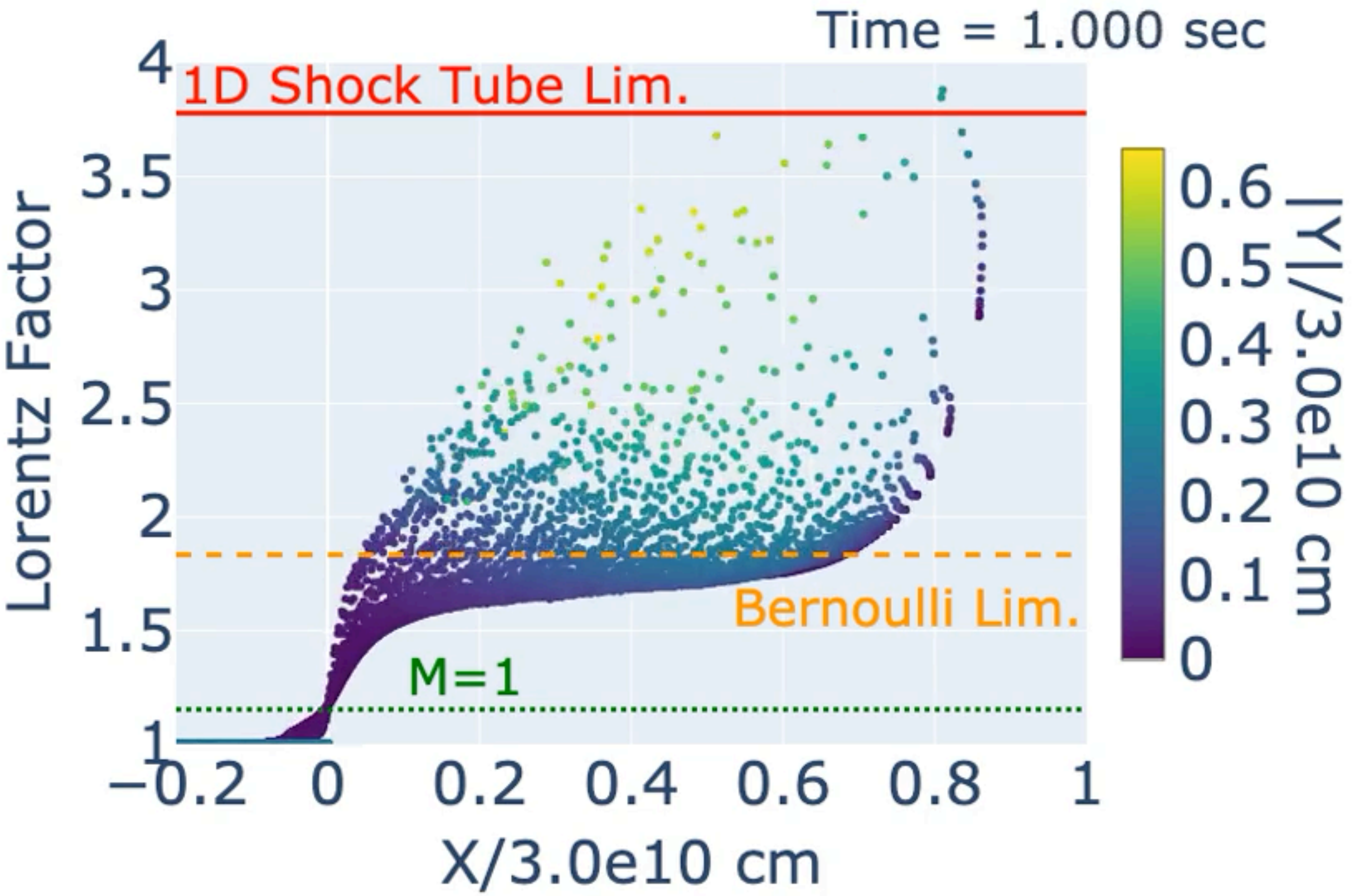
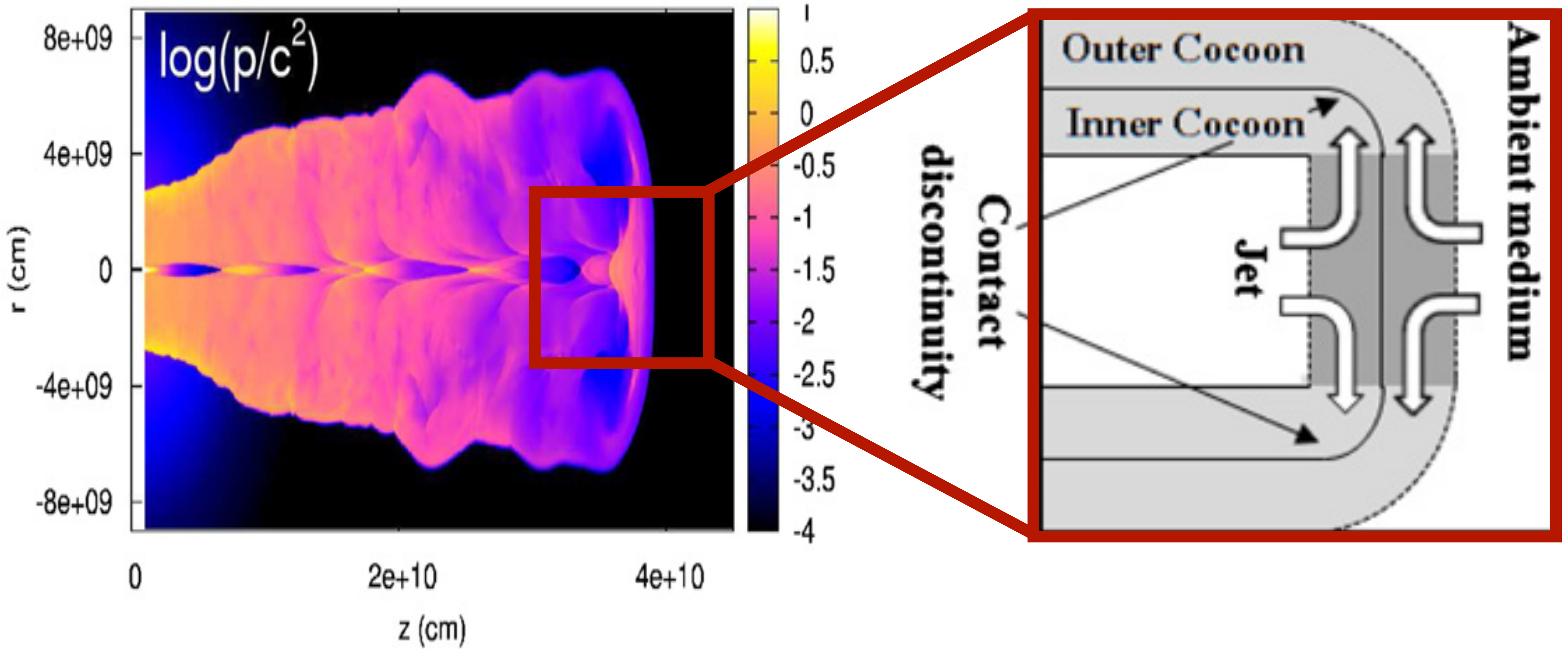




Research Background | Cocoo

This Study: Does High-Temperature, High-Density Gas Like the Outer Cocoon Behave as a Jet?

Cocoon Structure



- Collisions of relativistic jets create an **outer cocoon** of high-temperature, high-density gas.
- Its subsequent dynamical evolution is still largely **unexplored**.

Outline

- First Half — Special Relativistic Smoothed Particle Hydrorynamics

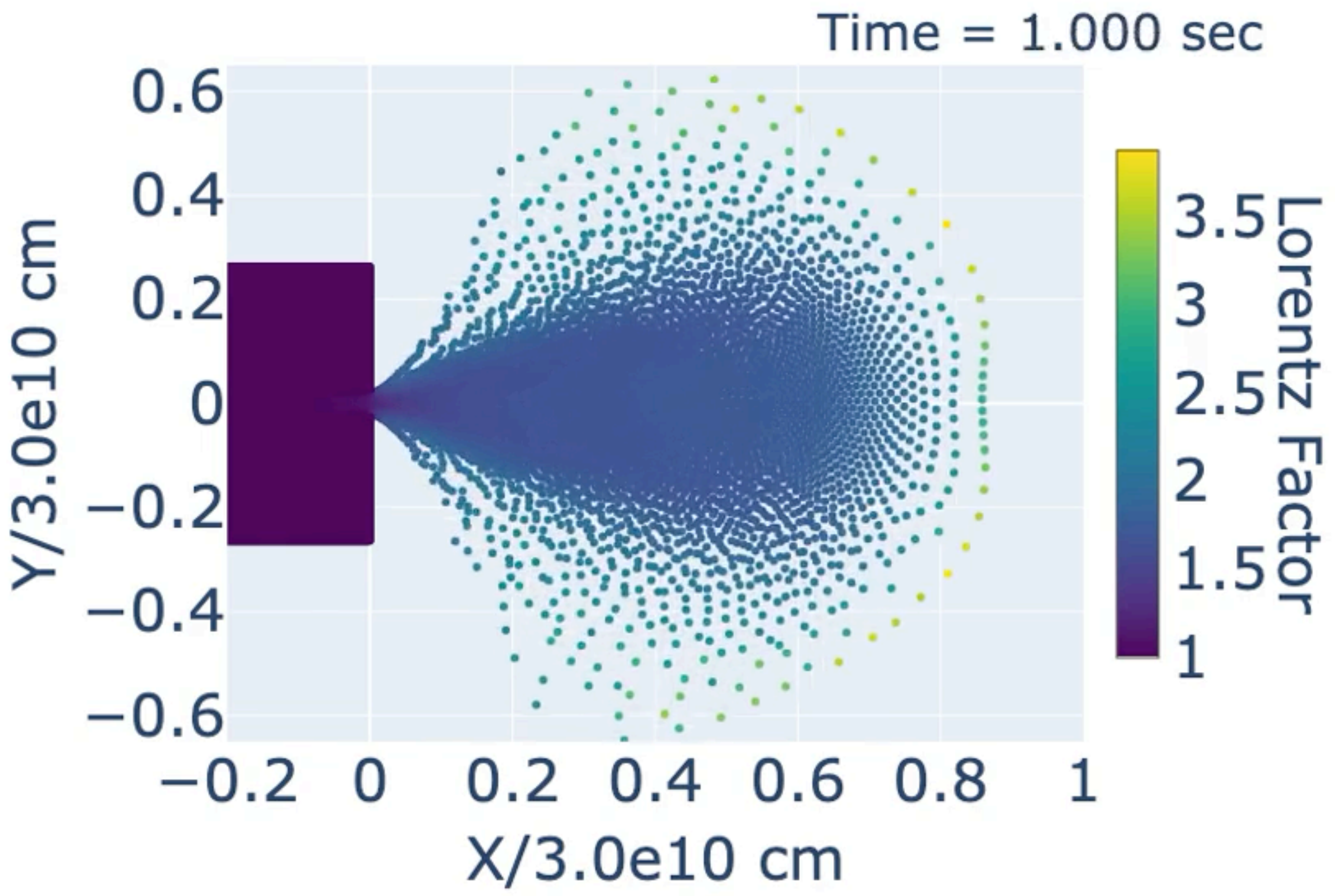
$$
\begin{cases}
N_i = \sum_j \nu_i W(\boldsymbol{x}_i - \boldsymbol{x}_j, h(\boldsymbol{x}_i)) \\
\nu_i \dot{\boldsymbol{S}}_i = -\sum_j P^*_{ij} V^2_{ij} \boldsymbol{F}_{ij} \\
\nu_i \dot{e}_i = -\sum_j P^*_{ij} \boldsymbol{v}^*_{ij} \cdot \boldsymbol{F}_{ij}
\end{cases}
$$

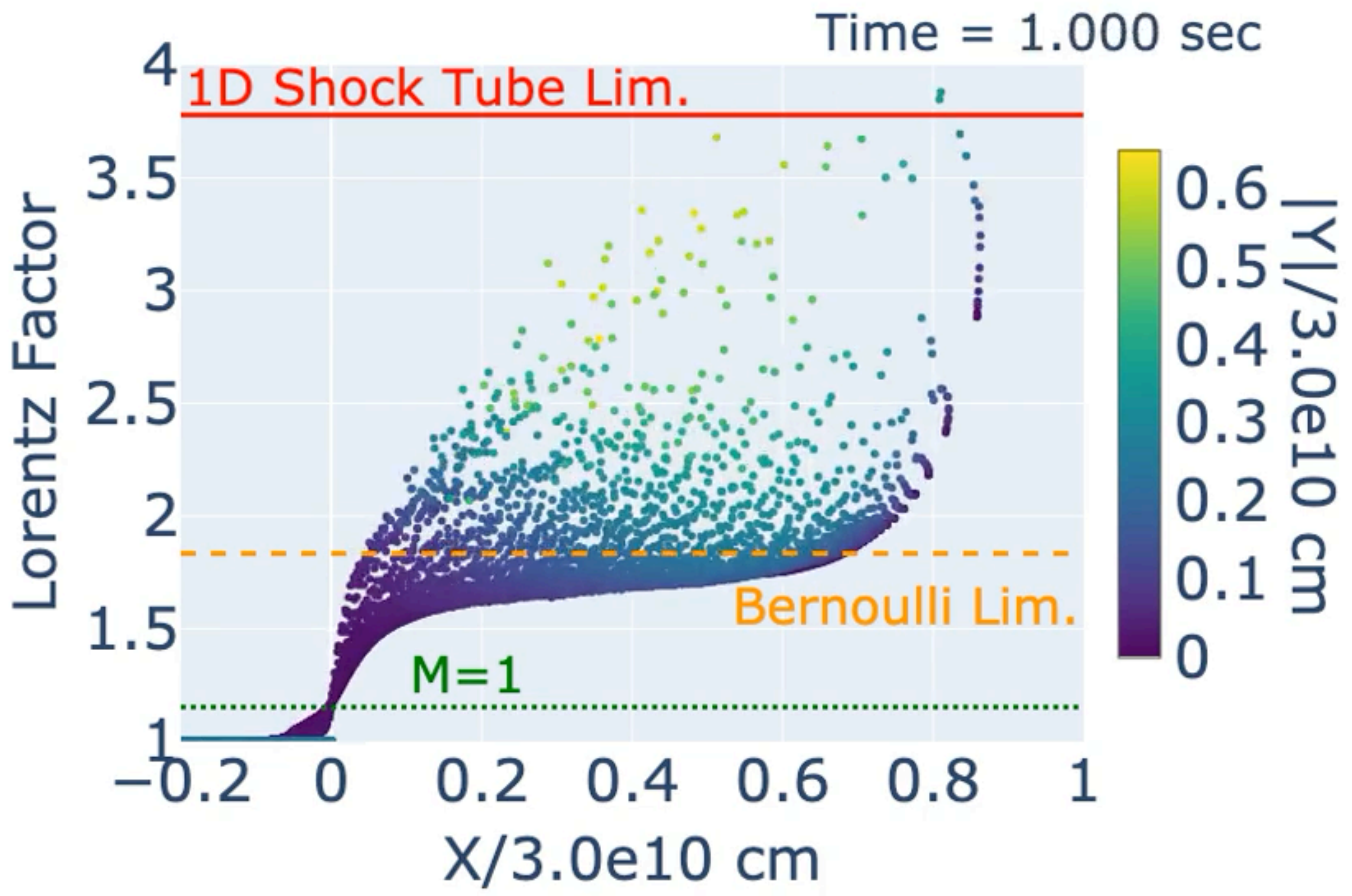
$$\boldsymbol{F}_{ij} := 2V^2_{ij}(h)\nabla_i W\left(\boldsymbol{x}_i - \boldsymbol{x}_j, \sqrt{2}h\right)$$

$$V^2_{ij}(h) := \int \frac{1}{N^2(\boldsymbol{x})}\left(\frac{\sqrt{2}}{h\sqrt{\pi}}\right)^d \times \exp\left[-\frac{2}{h^2}\left(\boldsymbol{x} - \frac{\boldsymbol{x}_i + \boldsymbol{x}_j}{2}\right)^2\right] d\boldsymbol{x}$$

Riemann Solver

- Second Half — Simulation of High-Speed Jet into Vacuum

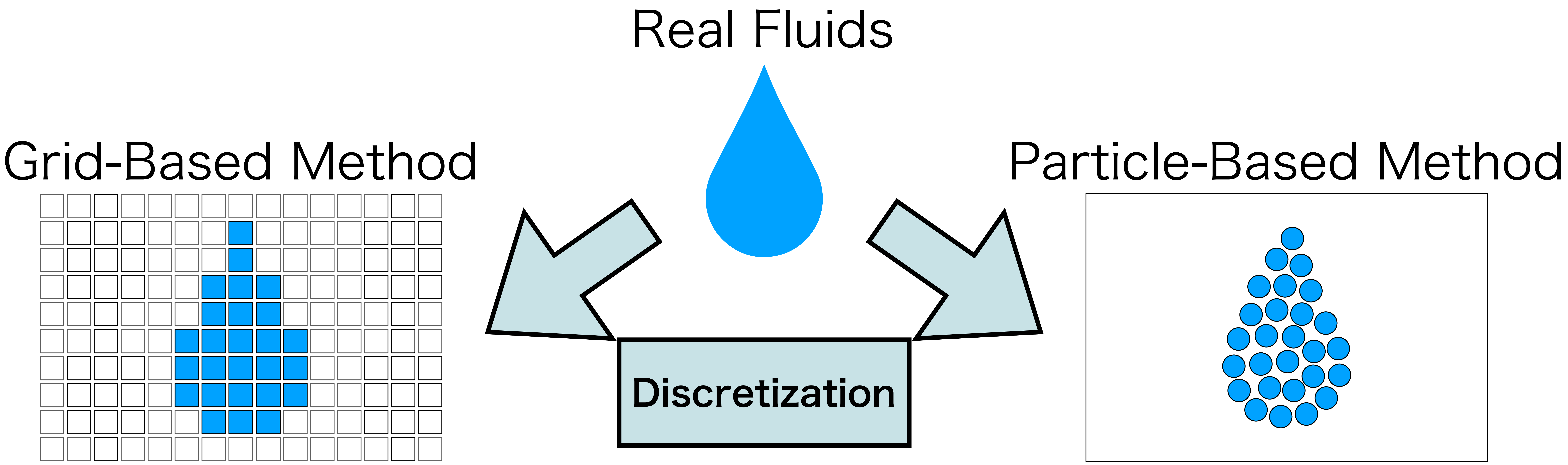




Various Numerical Hydrodynamics Methods

Due to limited computational resources, fluid calculations must be **discretized**.

Discretization is broadly classified into **grid-based methods** and **particle-based methods**.



Divides Space into a **Grid** and Computes Interactions Between Grid Cells
e.g., Finite Volume Method

Divides Fluid into **Particles** and Computes Their **Trajectories**
e.g., **SPH**

■ Definition of a Field

- Specific Volume @ Lab Frame

$$V_{\mathrm{p}}(\boldsymbol{x}) := \left[\sum_j W(\boldsymbol{x}-\boldsymbol{x}_j, h(\boldsymbol{x}))\right]^{-1}$$

■ Definition of Physical Quantities Carried by SPH Particles

- Number Density @ Lab Frame

$$N_i := N(\boldsymbol{x}_i) = \nu(\boldsymbol{x}_i) V_{\mathrm{p}}^{-1}(\boldsymbol{x}_i)$$

$\nu(\boldsymbol{x})$: Baryon Number Field

- Physical Quantities Other Than N_i

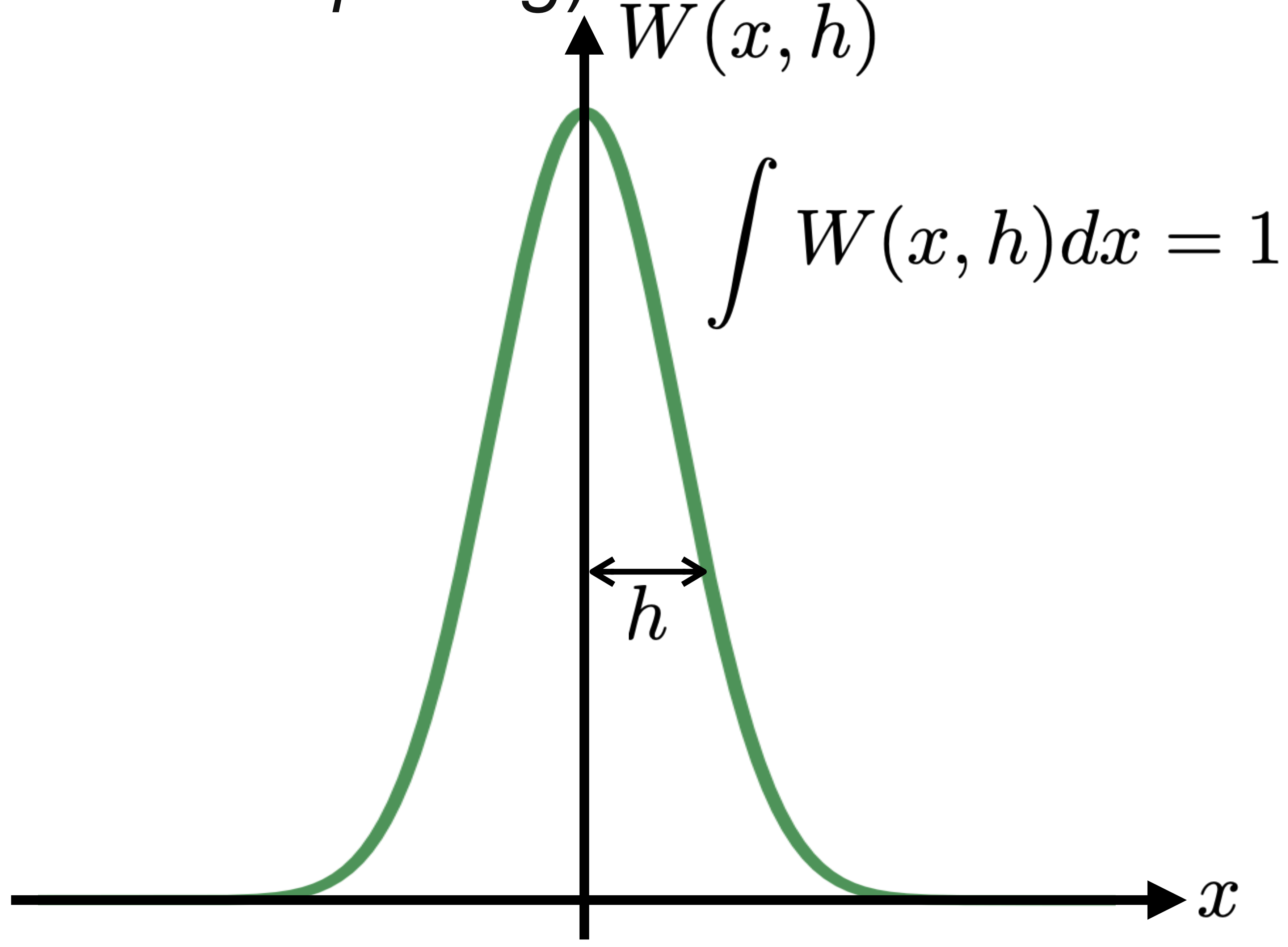
$$\nu_i f_i := \int \nu(\boldsymbol{x}) f(\boldsymbol{x}) W(\boldsymbol{x}-\boldsymbol{x}_i, h(\boldsymbol{x})) d\boldsymbol{x}$$

■ Kernel Function

$$W(\boldsymbol{x}, h) = \left(\frac{1}{h\sqrt{\pi}}\right)^d \exp\left(-\frac{|\boldsymbol{x}|^2}{h^2}\right)$$

d : Number of Spatial Dimensions

h : Smoothing Length
(Set to Approximately the Mean Particle Spacing)



Special Relativistic Godunov SPH Method | Basic Equations

- Eulerian Description

$$\begin{cases} \partial_t(\gamma\rho) = -\partial_i(\gamma\rho v^i) \\ \partial_t\left(\gamma\rho S^i\right) = -\partial_j\left(\gamma\rho S^i v^j + P\delta^{ij}\right) \\ \partial_t\left(\gamma\rho e\right) = -\partial_i\left[\gamma\rho v^i\left(\gamma H\right)\right] \end{cases}$$

$$\frac{d}{dt} := \partial_t + \boldsymbol{v}\cdot\nabla$$

- Lagrangian Description

$$\begin{cases} \dfrac{dNm}{dt} = -Nm\nabla\cdot\boldsymbol{v} \\ \dfrac{d\boldsymbol{S}}{dt} = -\dfrac{1}{Nm}\nabla P \\ \dfrac{de}{dt} = -\dfrac{1}{Nm}\nabla\cdot(P\boldsymbol{v}) \end{cases}$$

- **Basic Equations of the Special Relativistic Godunov SPH Method**

[**KK**, Inutsuka, Seno revised]

Convolution Integral Using Kernel Function

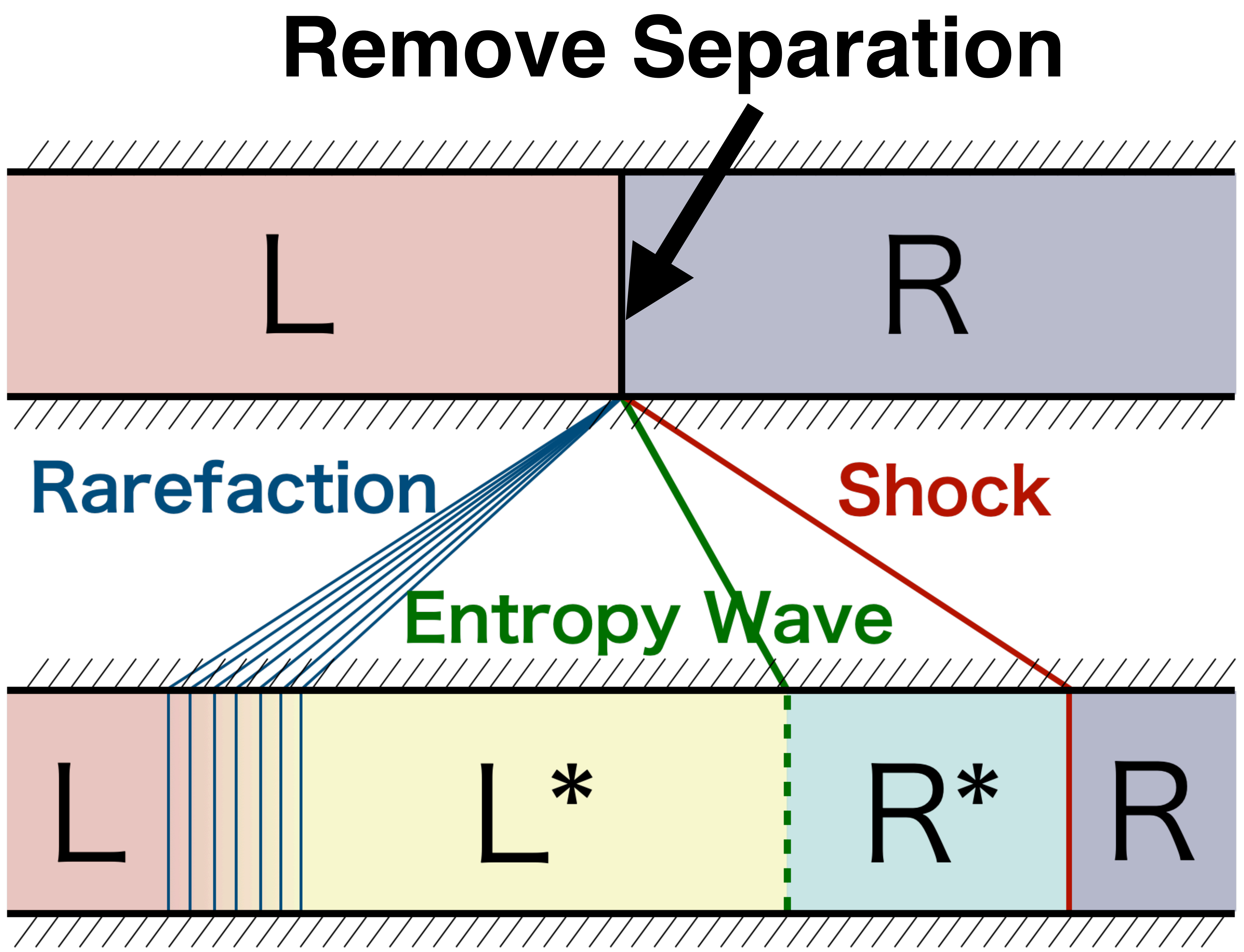
$$\begin{cases} N_i = \sum_j \nu_i W(\boldsymbol{x}_i - \boldsymbol{x}_j, h(\boldsymbol{x}_i)) \\ \nu_i\dot{\boldsymbol{S}}_i = -\sum_j P^*_{ij} V^2_{ij}\boldsymbol{F}_{ij} \\ \nu_i\dot{e}_i = -\sum_j P^*_{ij}\boldsymbol{v}^*_{ij}\cdot\boldsymbol{F}_{ij} \end{cases}$$

$$\boldsymbol{F}_{ij} := 2V^2_{ij}(h)\nabla_i W\left(\boldsymbol{x}_i - \boldsymbol{x}_j, \sqrt{2}h\right)$$

$$V^2_{ij}(h) := \int \frac{1}{N^2(\boldsymbol{x})}\left(\frac{\sqrt{2}}{h\sqrt{\pi}}\right)^d \times \exp\left[-\frac{2}{h^2}\left(\boldsymbol{x} - \frac{\boldsymbol{x}_i + \boldsymbol{x}_j}{2}\right)^2\right] d\boldsymbol{x}$$

Riemann Solver

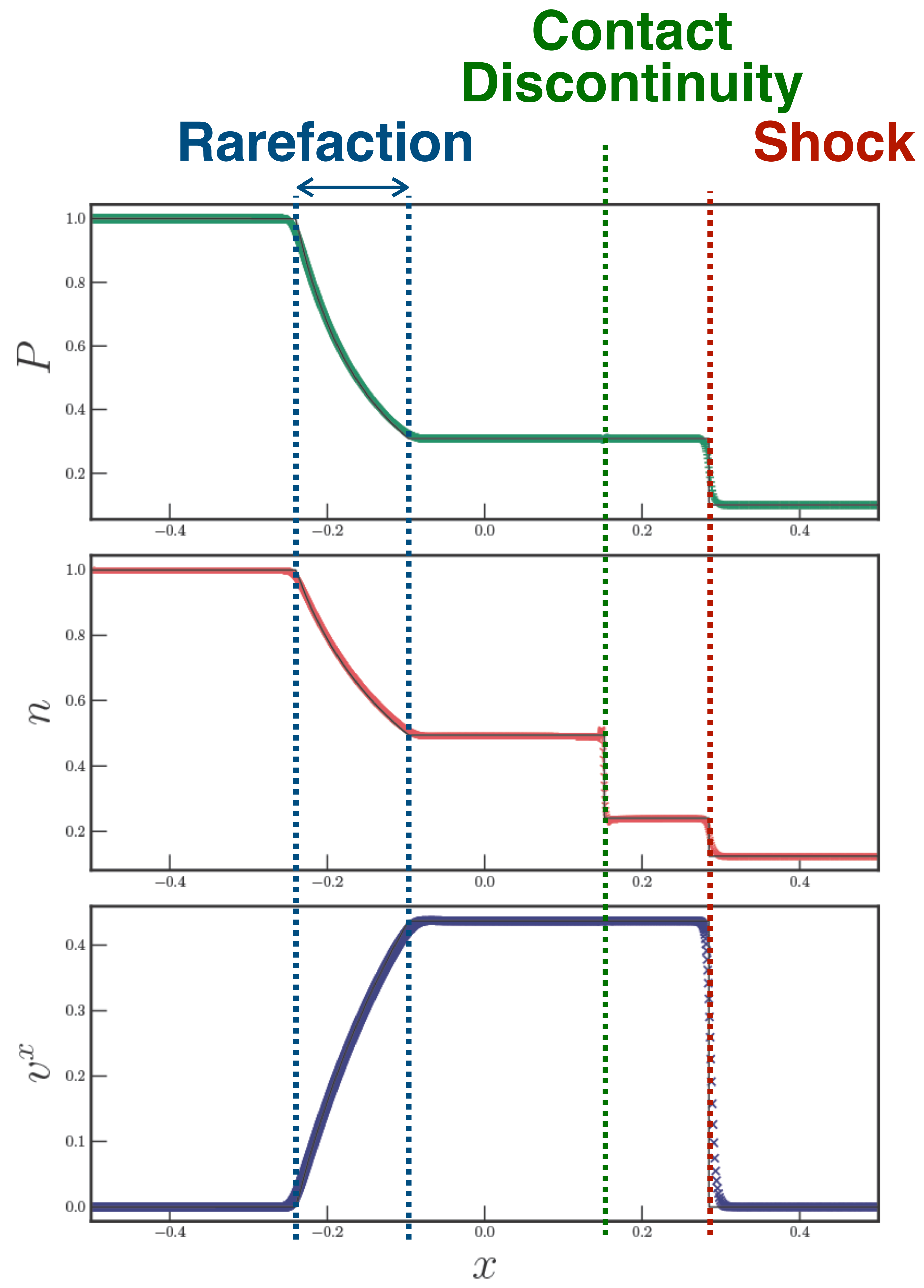
Example Spacetime Diagram



Riemann Problem Applications:

- Numerical Computation
  → **Godunov Method**
- **Test Calculations**

Test Calculation Result



Compared with analytical solution
→ high accuracy

Outline

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$$
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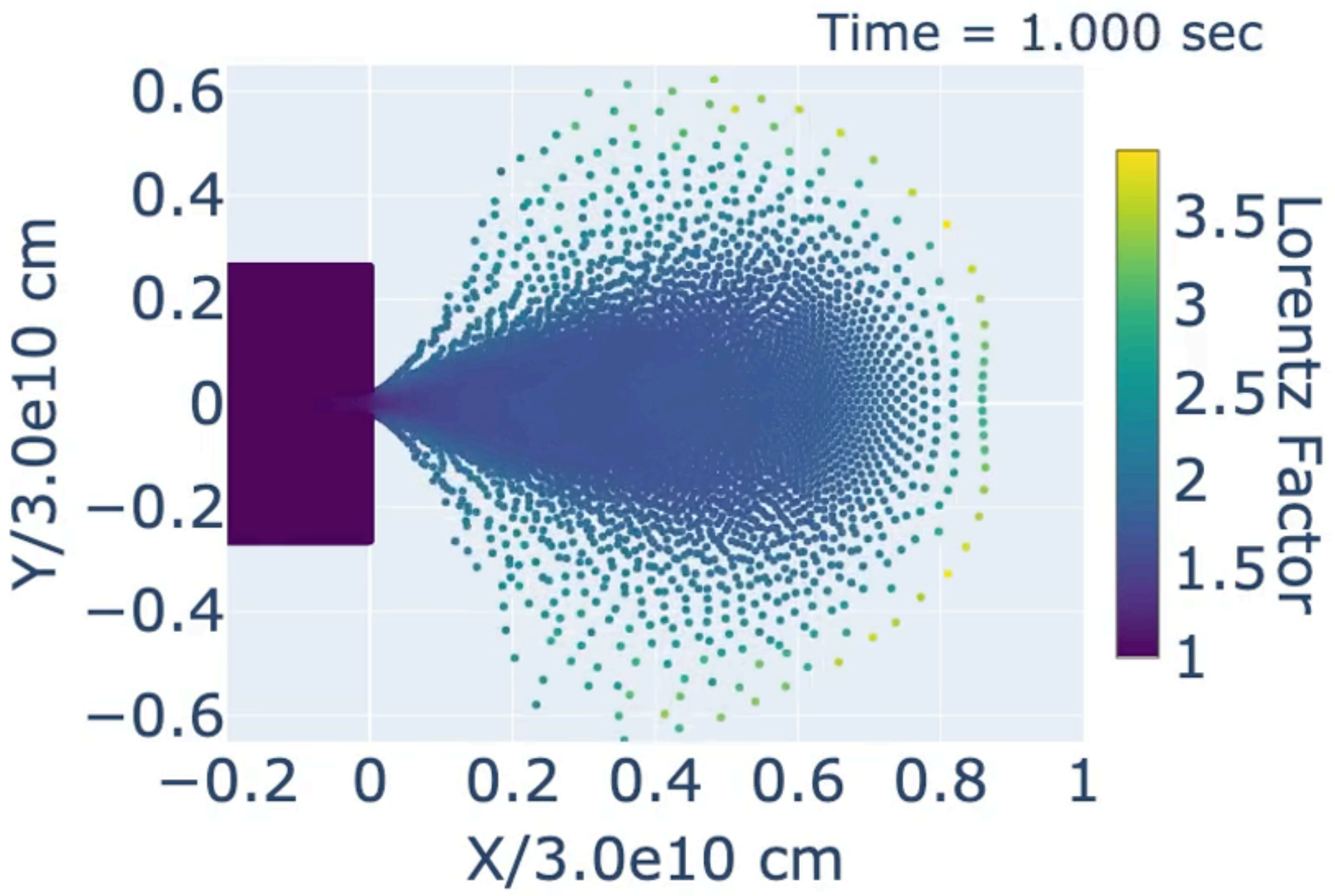
$$

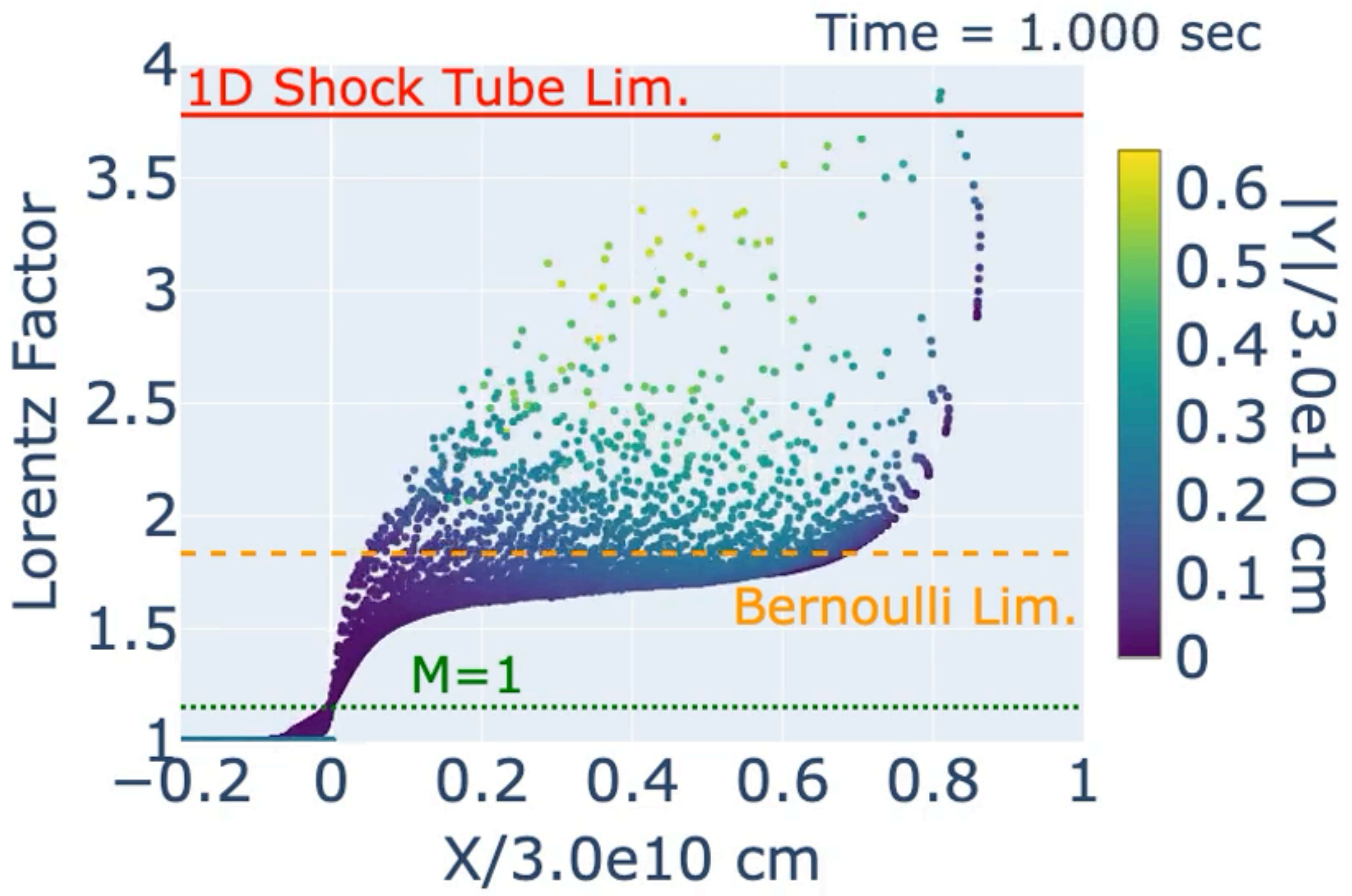
$$
\boldsymbol{F}_{ij} := 2V_{ij}^2(h)\nabla_i W\left(\boldsymbol{x}_i - \boldsymbol{x}_j, \sqrt{2}h\right)
$$

$$
V_{ij}^2(h) := \int \frac{1}{N^2(\boldsymbol{x})}\left(\frac{\sqrt{2}}{h\sqrt{\pi}}\right)^d \times \exp\left[-\frac{2}{h^2}\left(\boldsymbol{x} - \frac{\boldsymbol{x}_i + \boldsymbol{x}_j}{2}\right)^2\right] d\boldsymbol{x}
$$

Riemann Solver

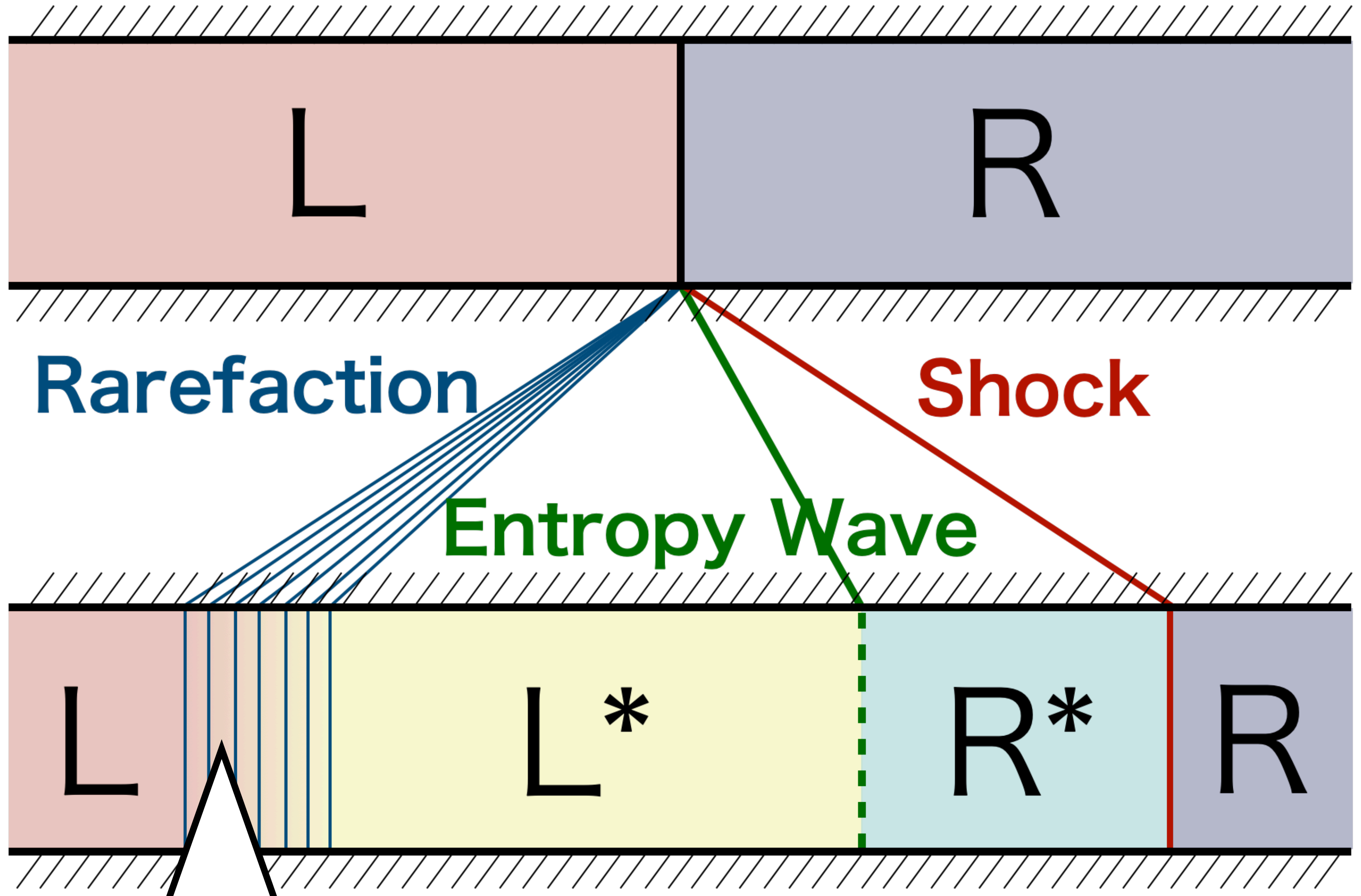
- Second Half — Simulation of High-Speed Jet into Vacuum



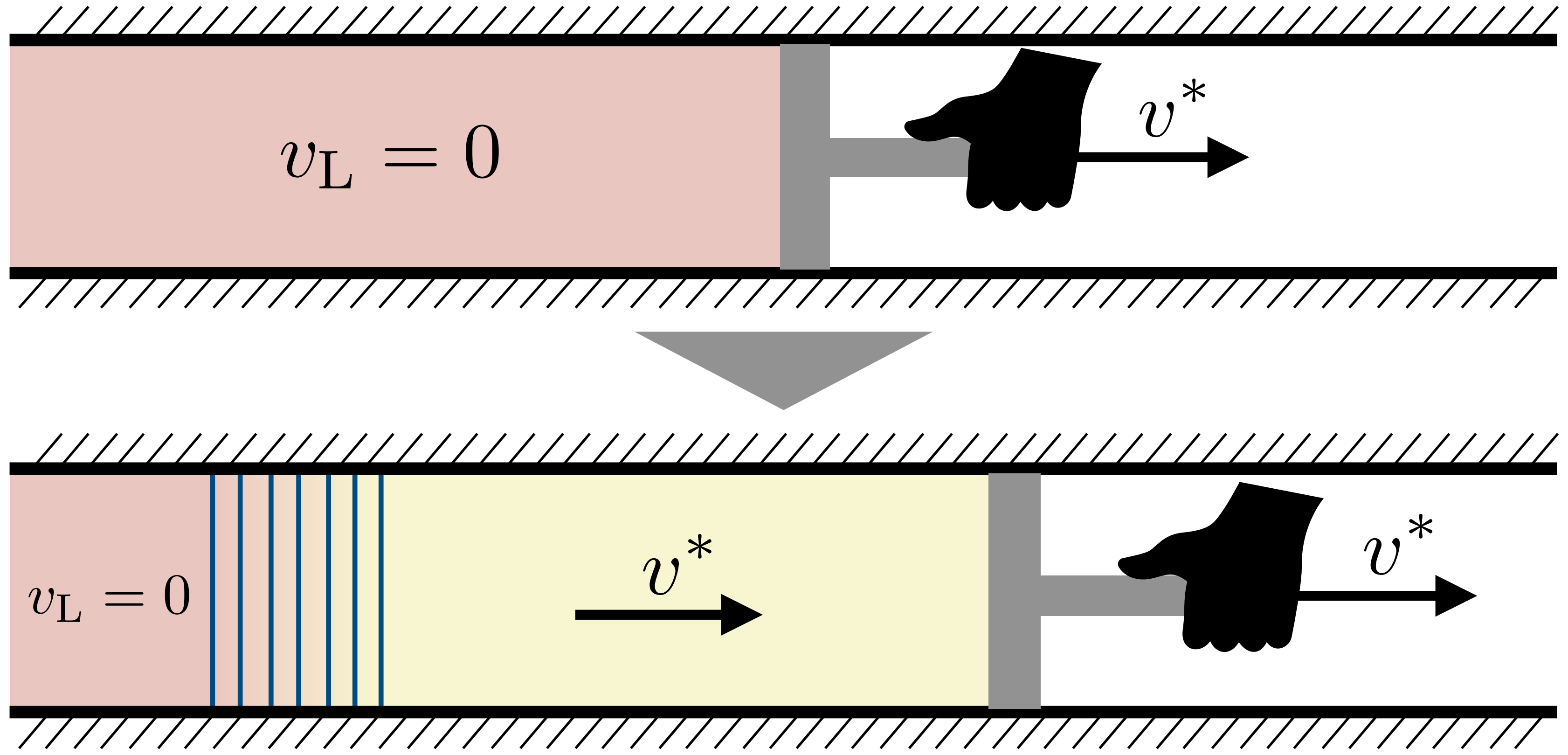


Acceleration Mechanisms

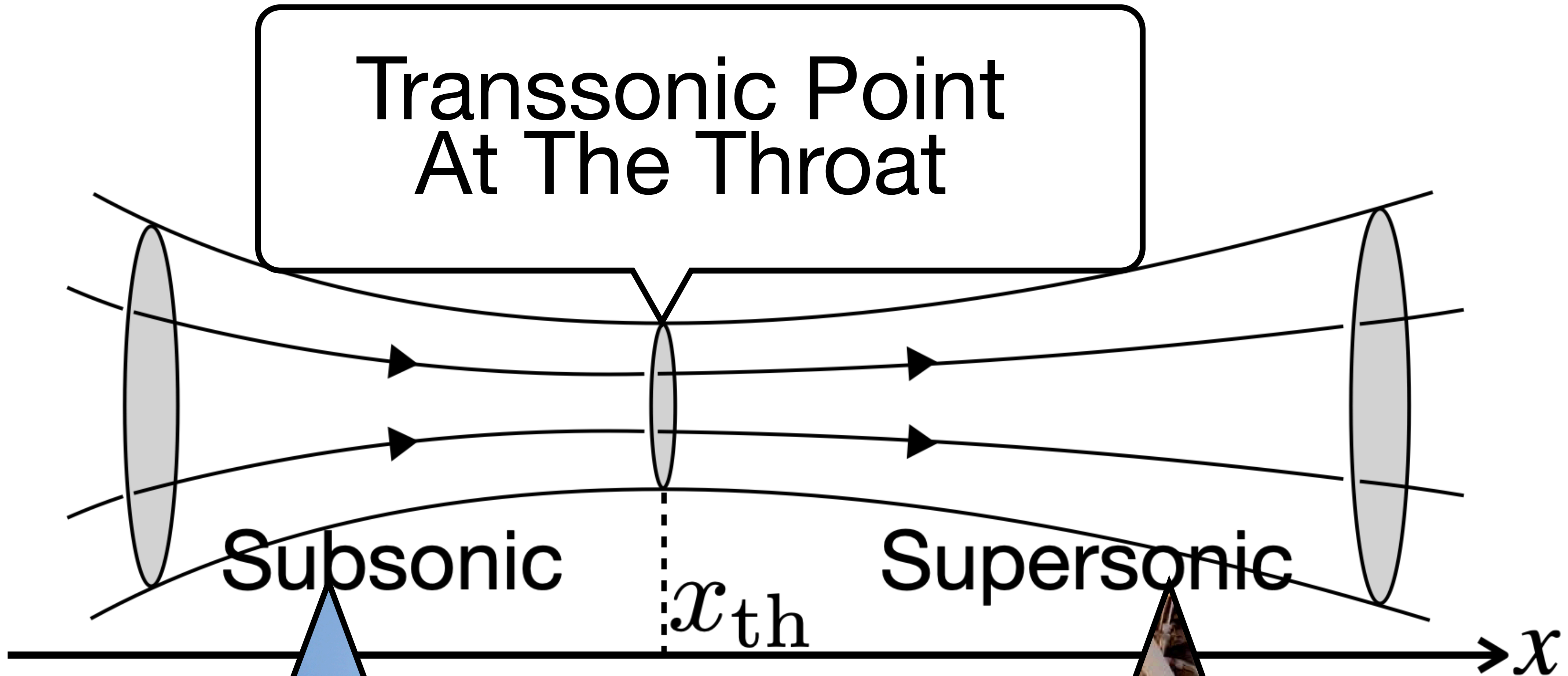
- Unsteady Acceleration — Shock Tube Problem

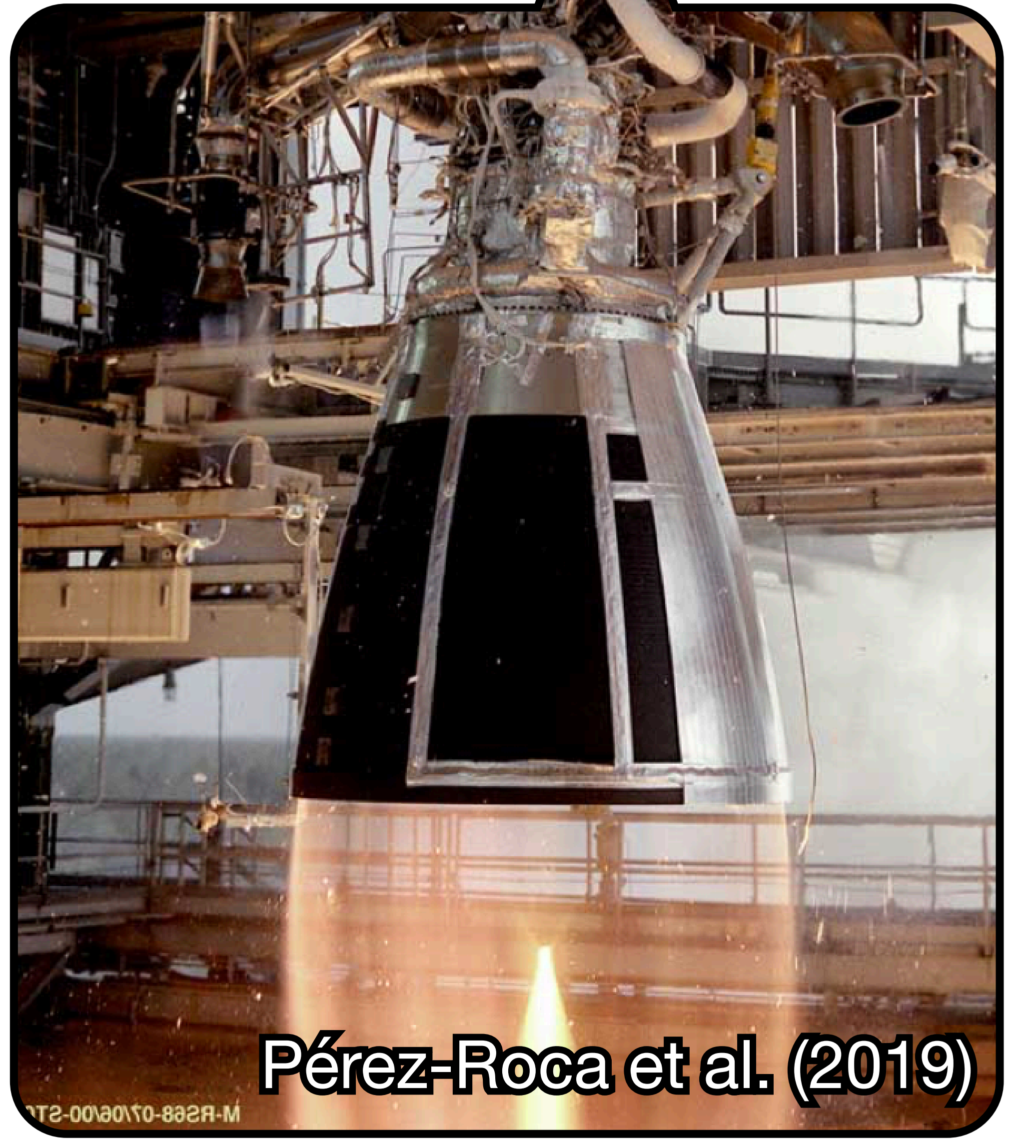


Similar to Acceleration by Pulling a Piston



- Steady Acceleration — De Laval Nozzle

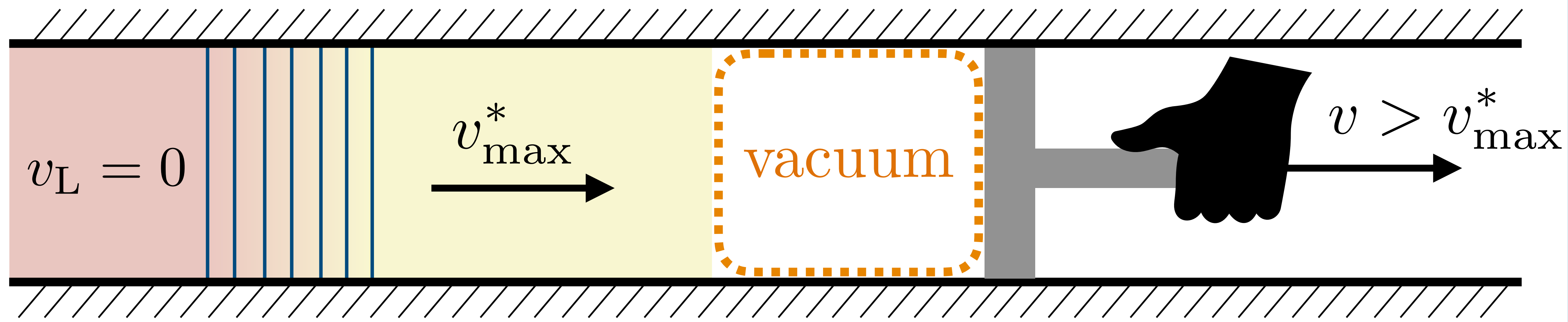




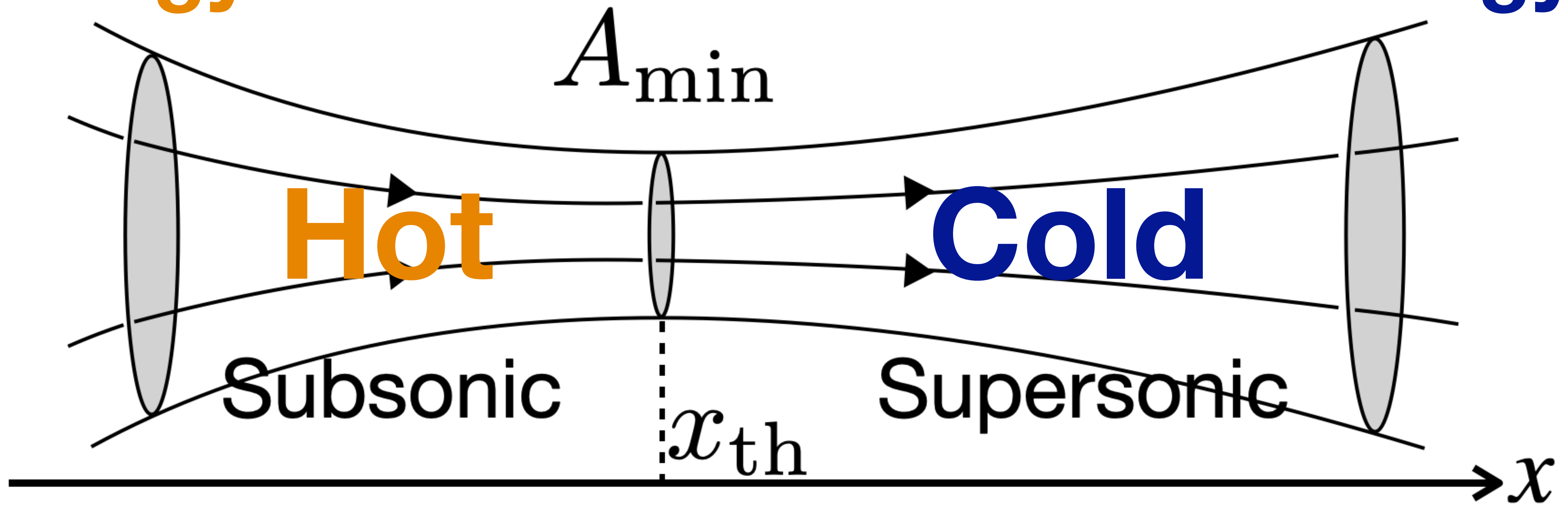
Pérez-Roca et al. (2019)

Comparison of Maximum Acceleration

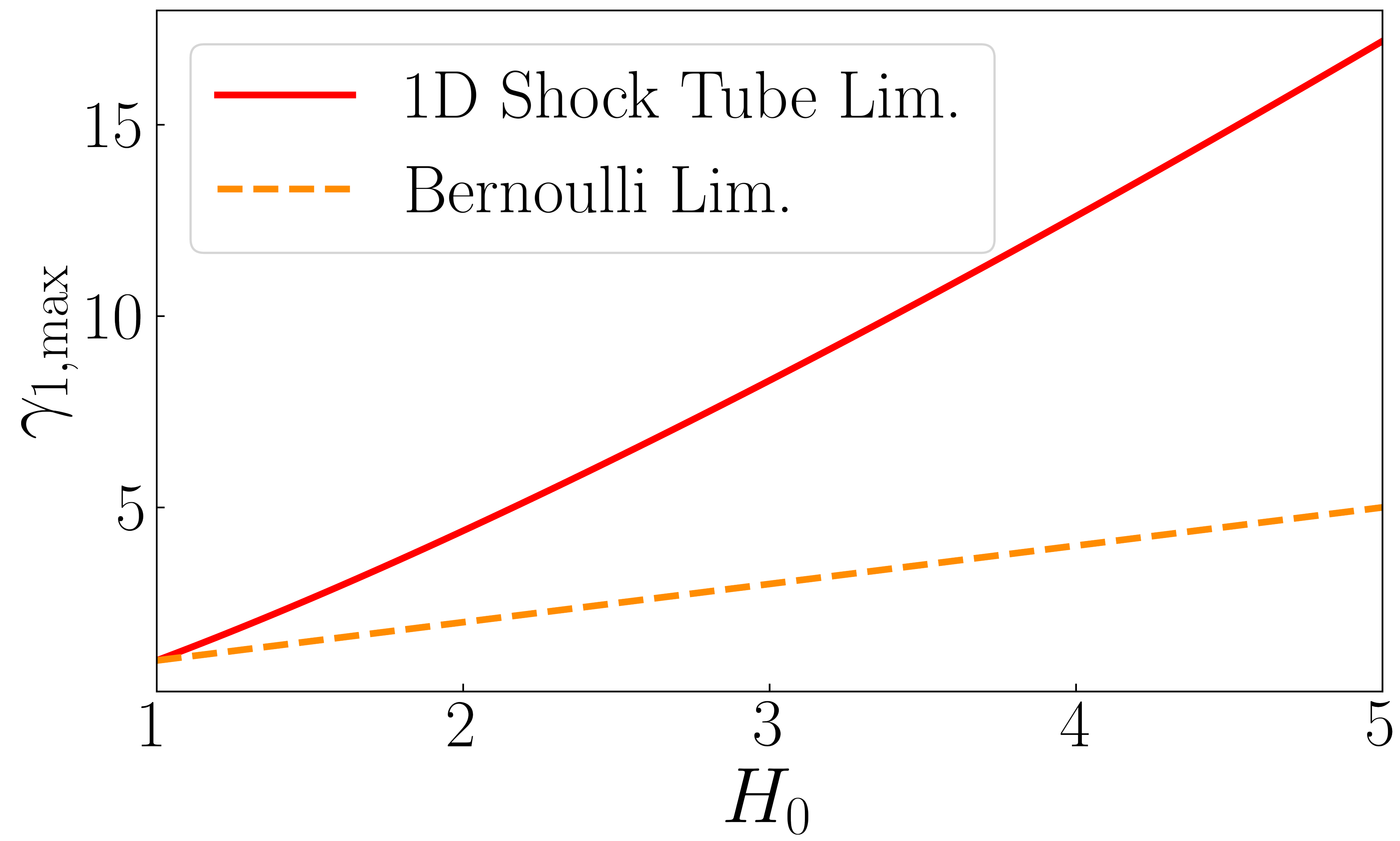
- "Acceleration Limit" in the 1D Shock Tube
  - Maximum Velocity is Achieved When Expanding into a **Vacuum**



- Steady Acceleration — De Laval Nozzle
  - **Maximum Velocity** When **All Thermal Energy** Is Converted into **Kinetic Energy**

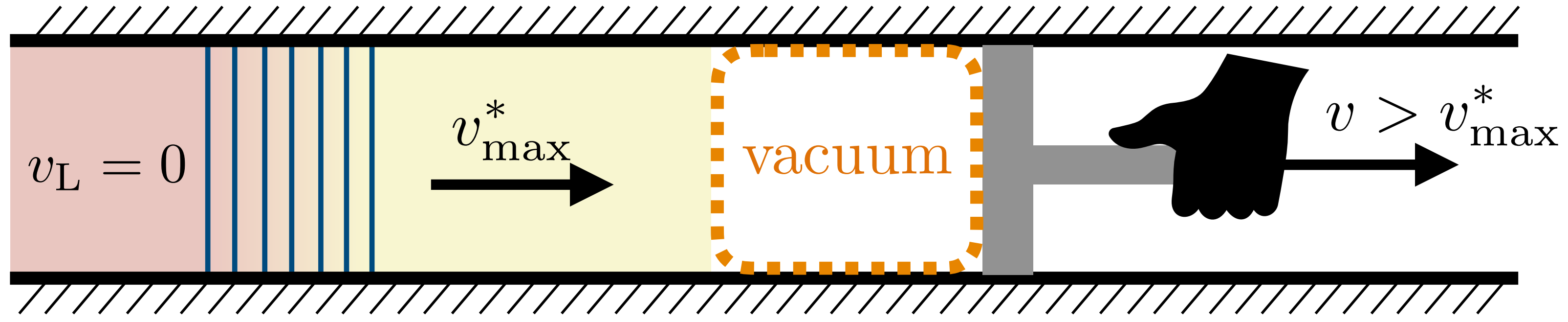


- In both cases, maximum acceleration occurs when expanded to **vacuum**.
- The Lorentz factor after acceleration is higher in the shock tube than in the De Laval nozzle.

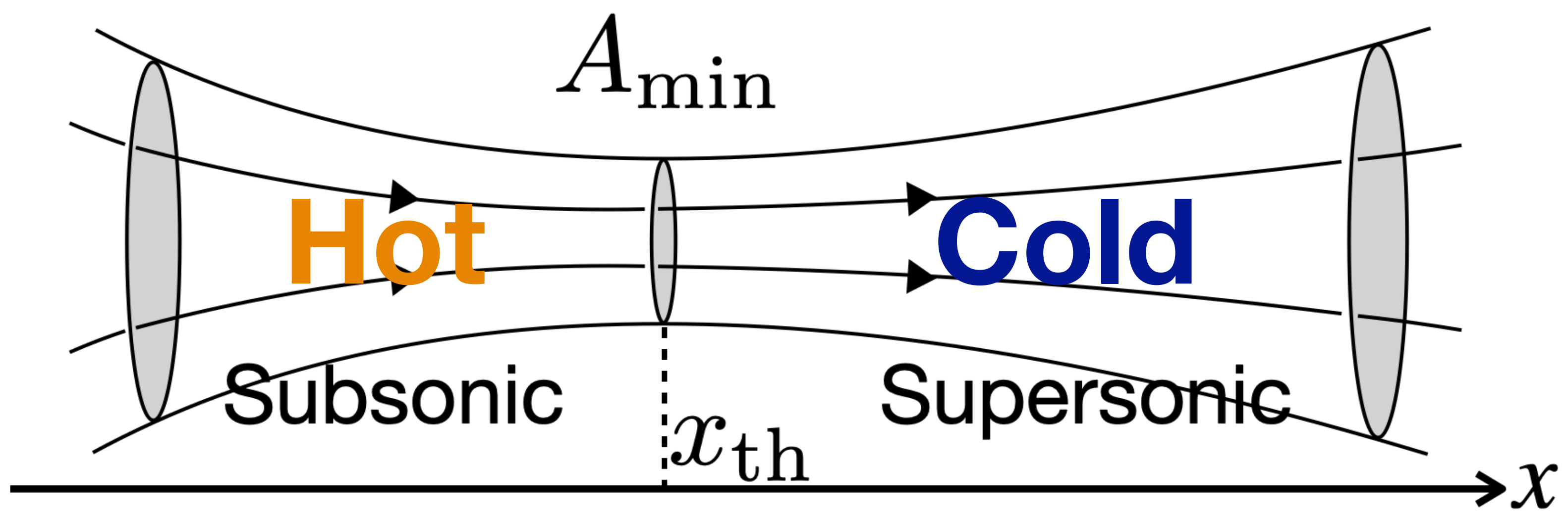


Challenges of Vacuum Computation in Numerical Simulations

- Acceleration Limit Derived from the 1D Shock Tube



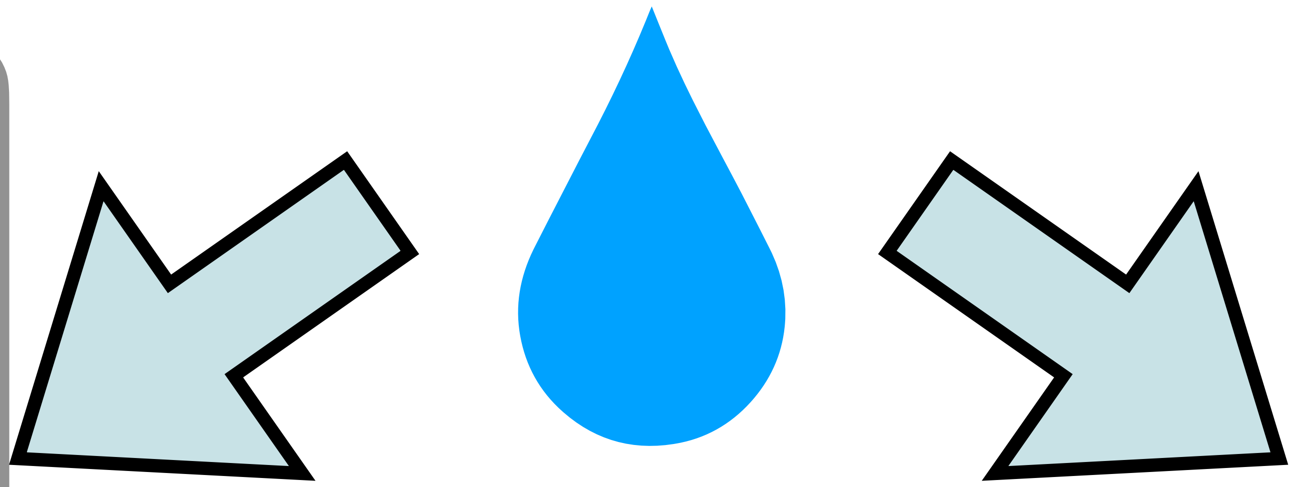
- Acceleration Limit Derived from Bernoulli's Theorem



In both cases, handling "**vacuum**" is essential to achieve maximum acceleration.

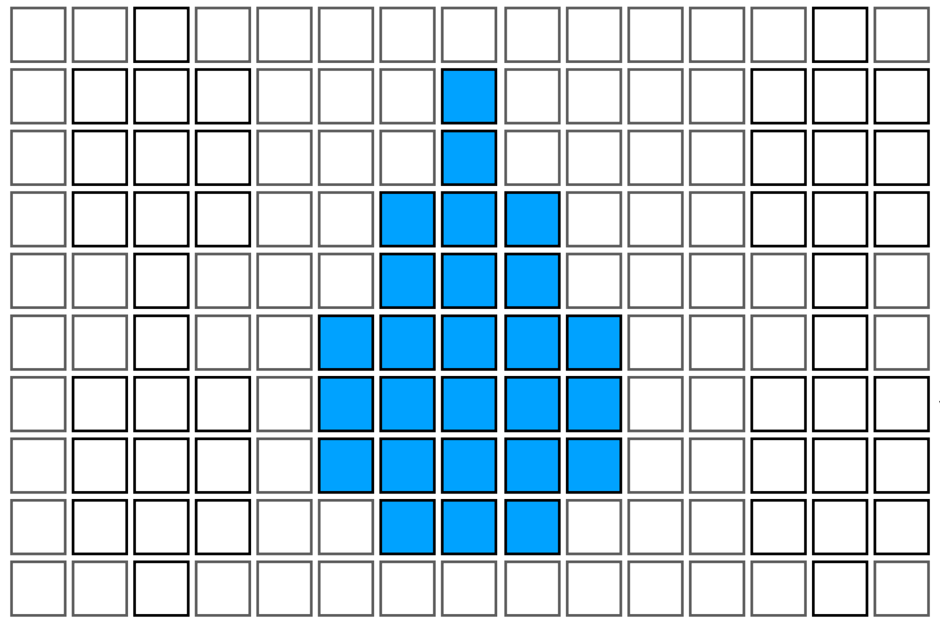
- Types of Numerical Fluid Dynamics Methods

Real Fluids



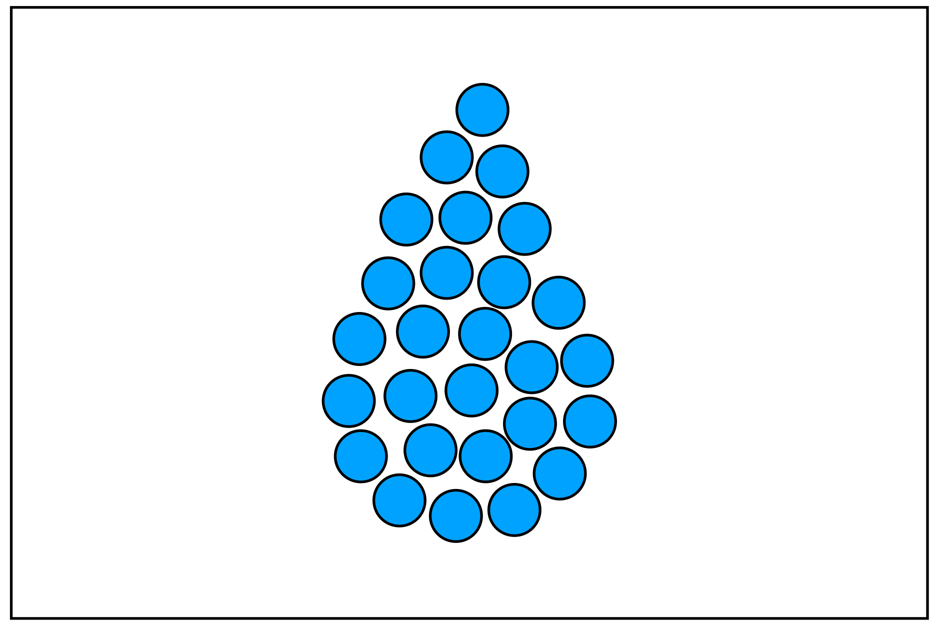
Discretization

Grid-Based Method



Unsuitable for Handling Vacuum

Particle-Based Method

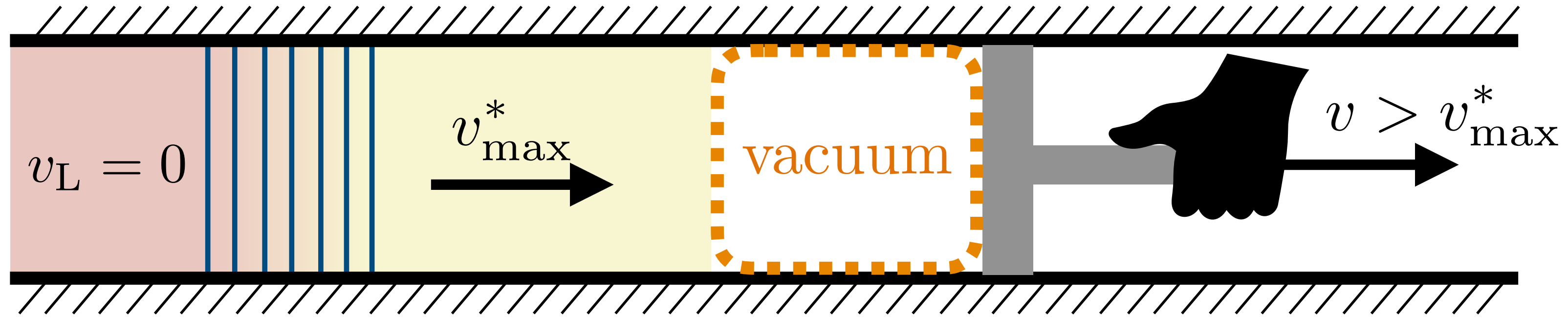


Advantageous Method for Handling **Vacuum**

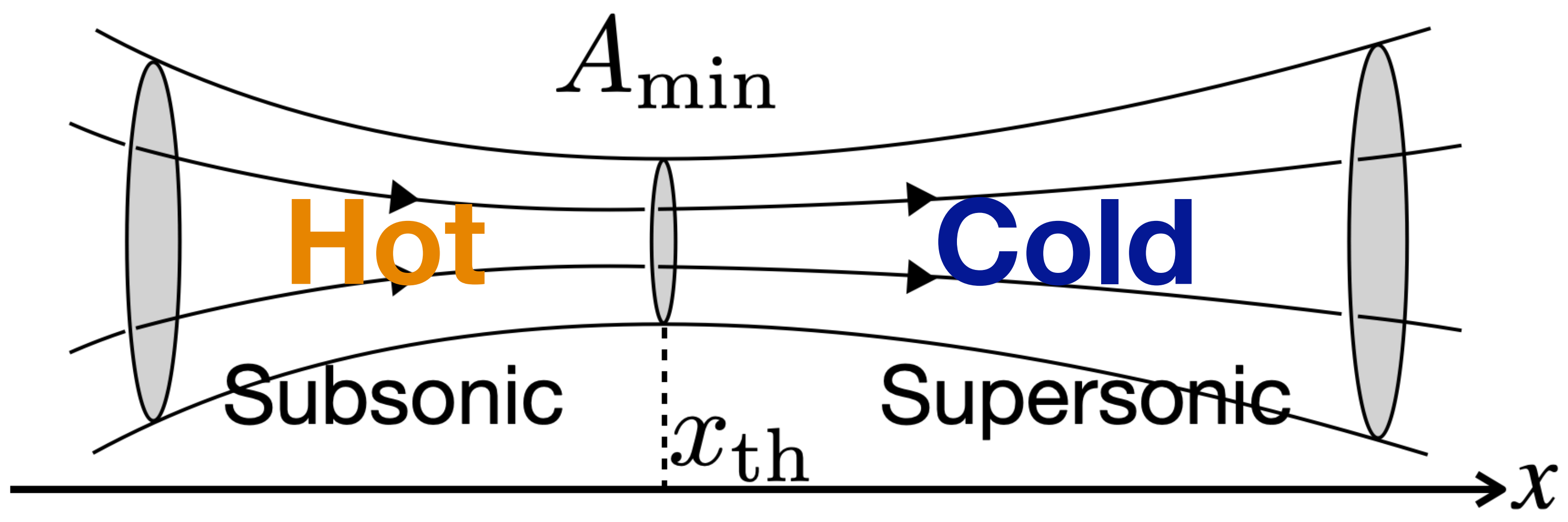
Analysis Based on the SPH Method

Challenges of Vacuum Computation in Numerical Simulations

- Acceleration Limit Derived from the 1D Shock Tube



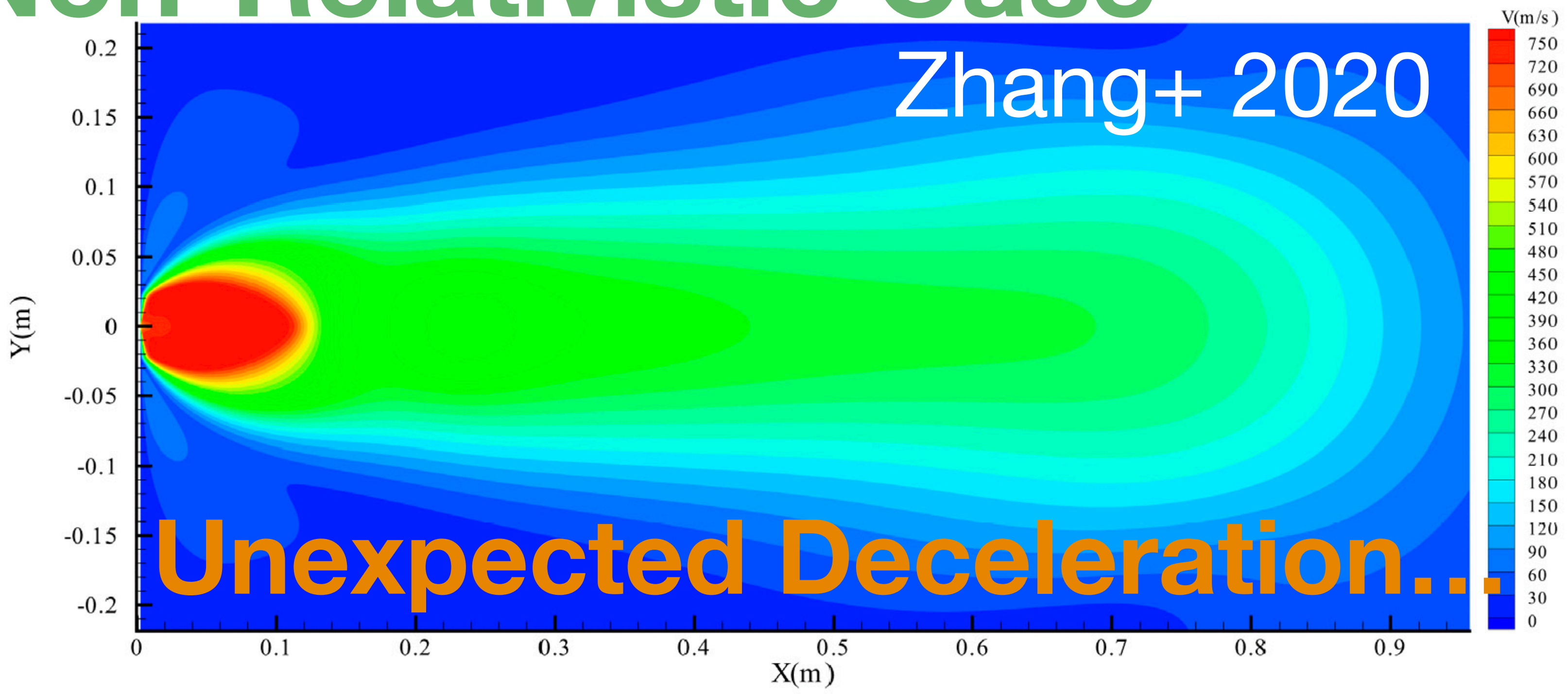
- Acceleration Limit Derived from Bernoulli's Theorem



In both cases, handling"**vacuum**" is essential to achieve maximum acceleration.

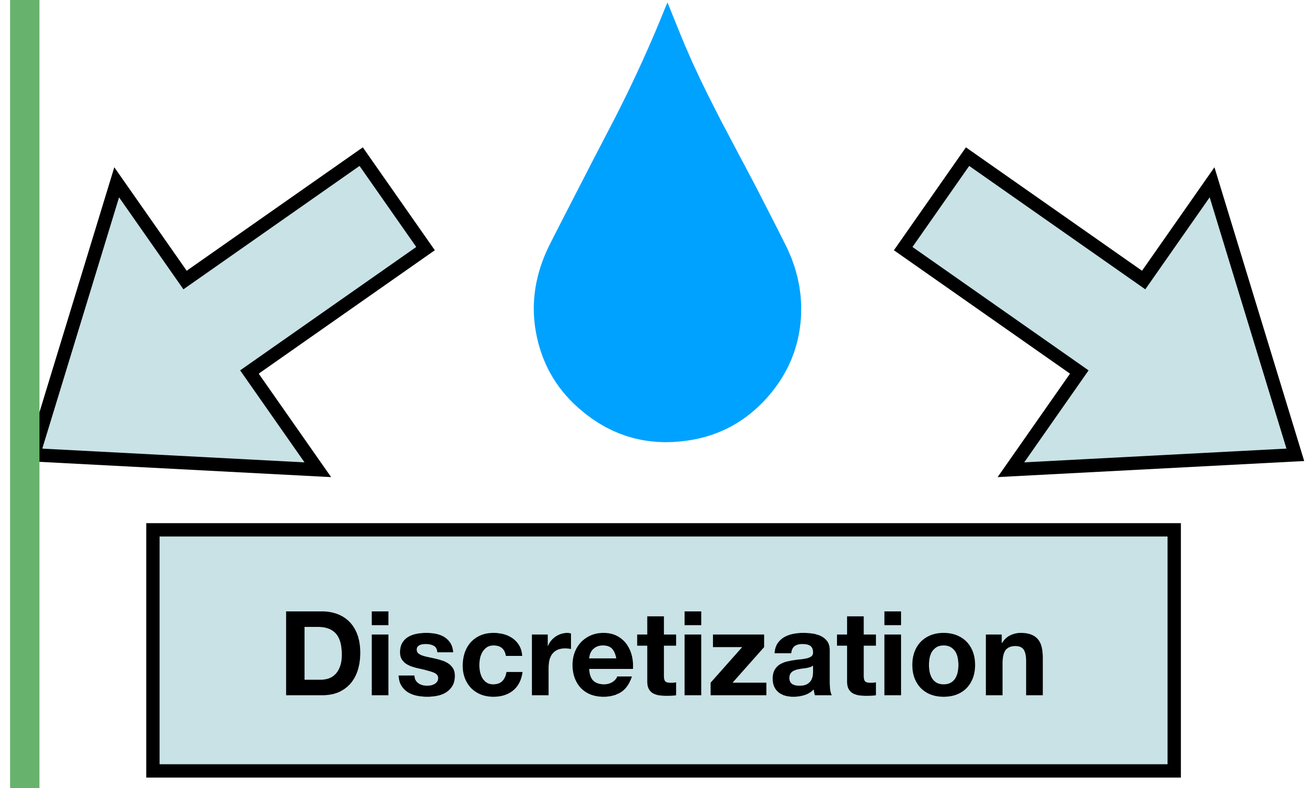
■ Types of Numerical Fluid Dynamics Methods

Non-Relativistic Case

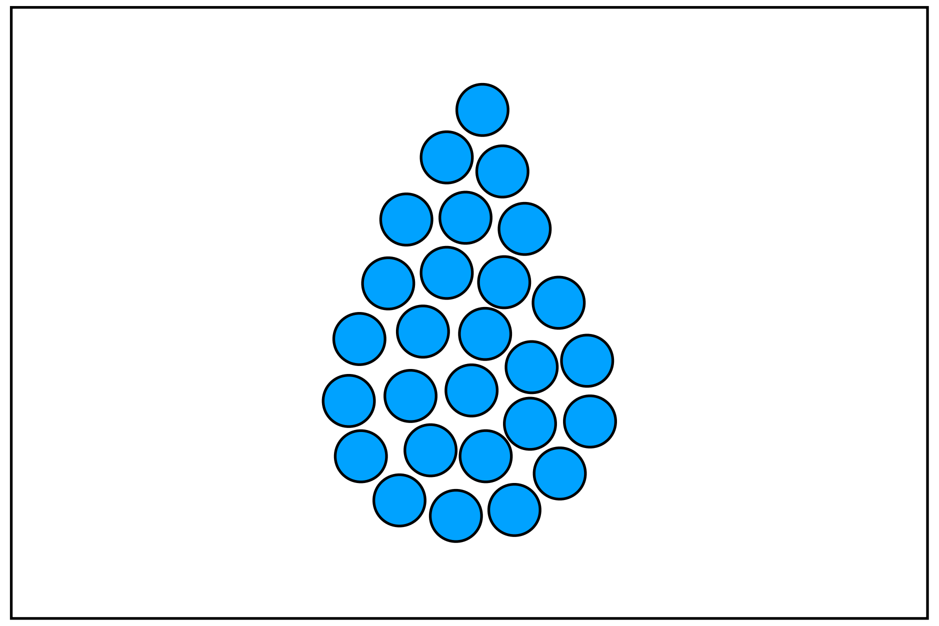


Even the Non-Relativistic Case Is Not Properly Computed

Real Fluids



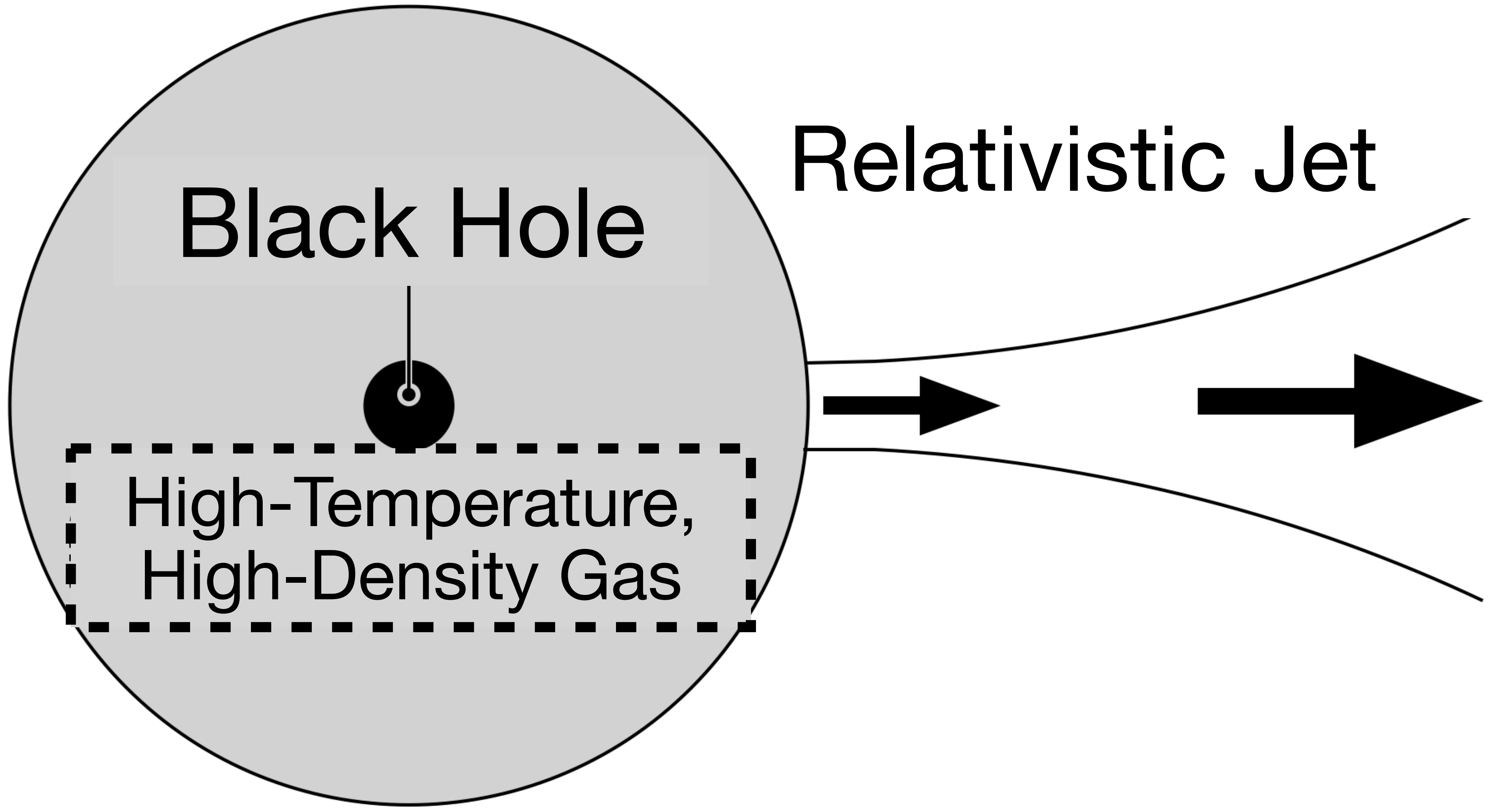
Particle-Based Method



Advantageous Method for Handling **Vacuum**

Analysis Based on the SPH Method

Simulation Model



Assuming gas falls to the Schwarzschild radius r_s and all gravitational potential energy is converted into thermal energy

$$\frac{\frac{GMm}{r_s}}{mc^2} = \frac{1}{2} = \frac{u}{c^2}$$

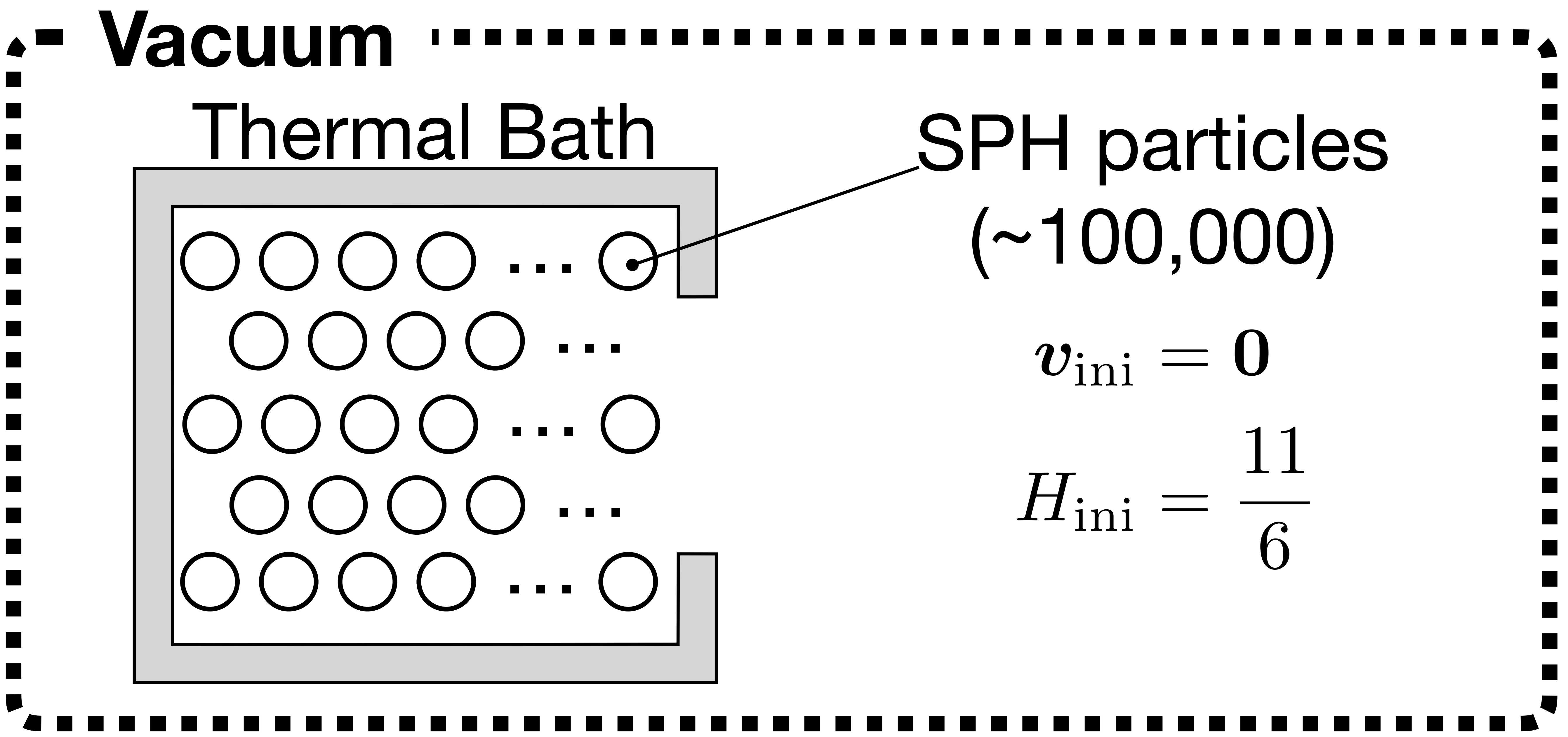
$$\Rightarrow H = \frac{11}{6}$$

$$r_s := \frac{2GM}{c^2}$$

$$H := 1 + \frac{u}{c^2} + \frac{P}{\rho c^2}$$

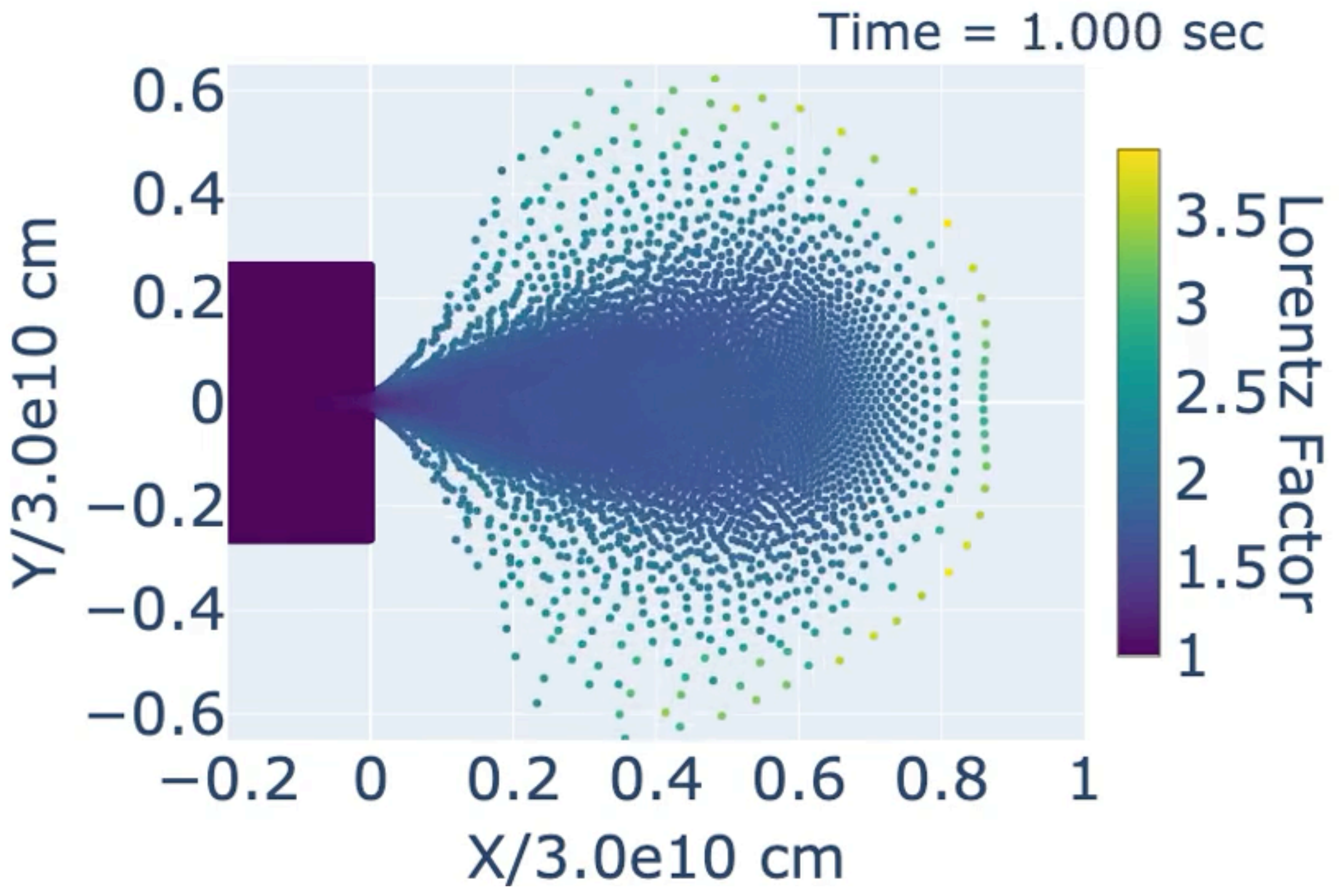
$$P = (\Gamma - 1)\rho u$$

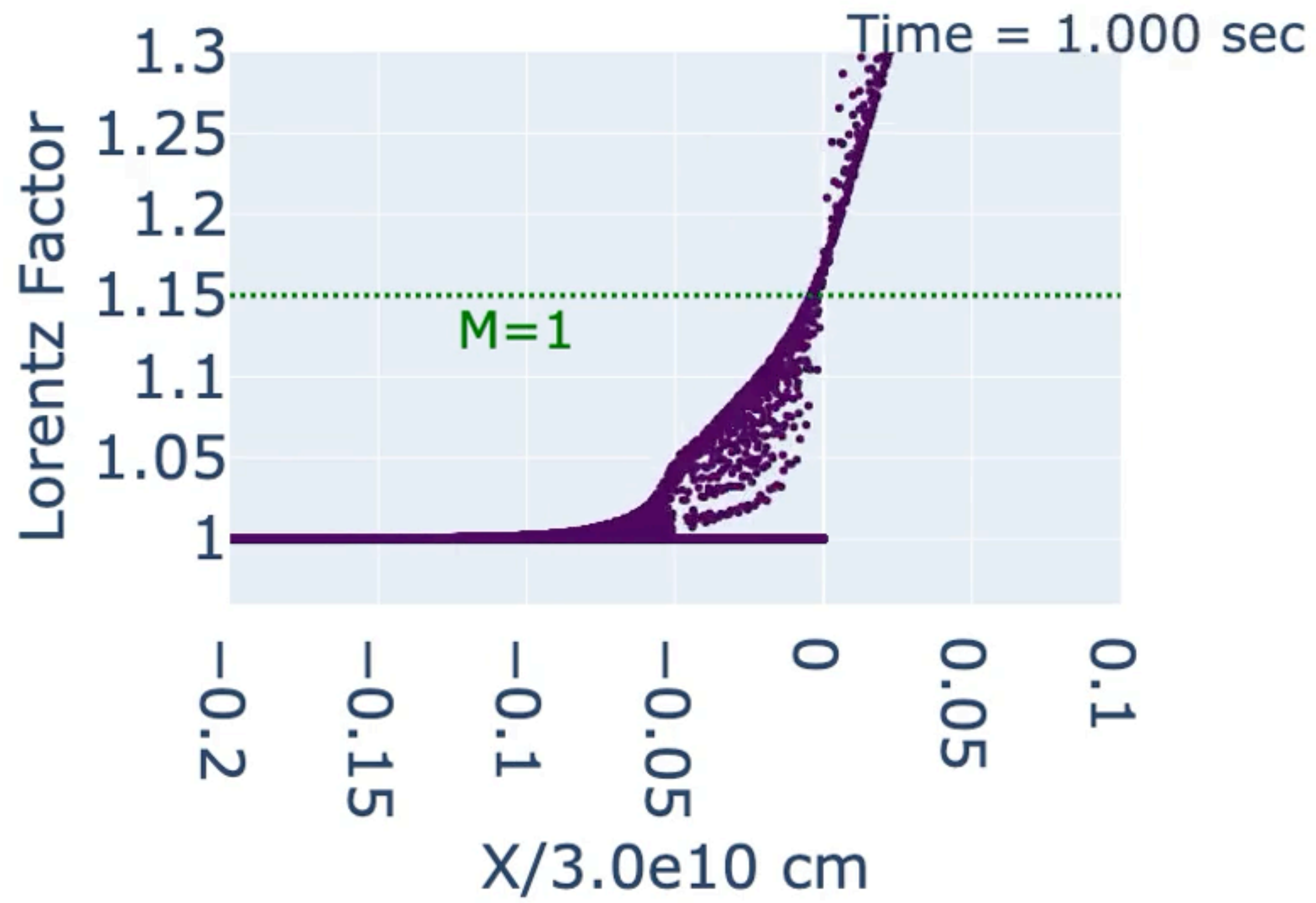
Initial Conditions (2D Simulation)



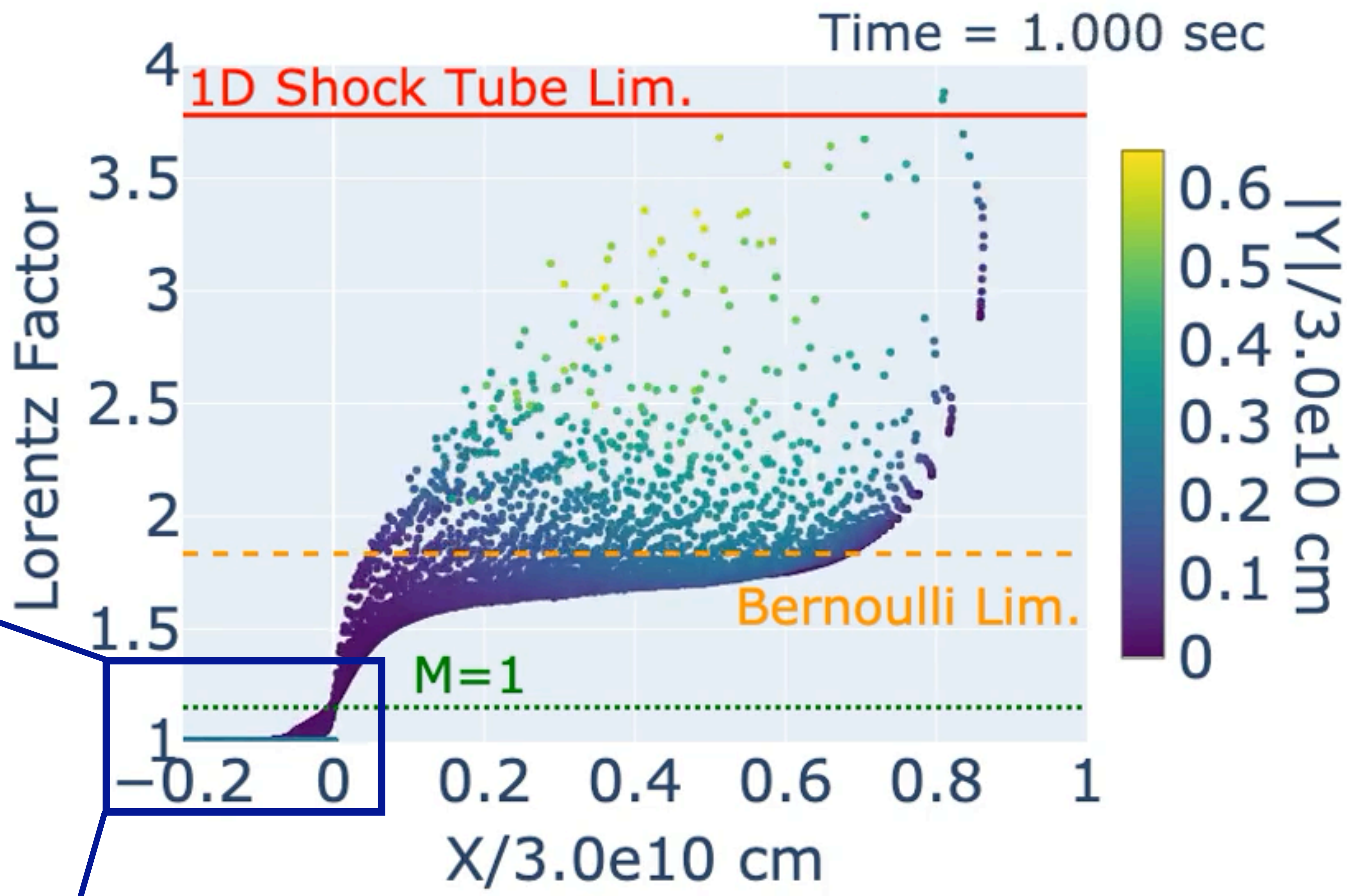
- Sound Speed — $\mathcal{M} = 1 \Leftrightarrow \gamma_{\rm th} \approx 1.15$
  Steady-State Limit — $\gamma_{\rm Ber,max} = H_{\rm ini} = 11/6$
  1D Shock Tube Limit — $\gamma_{\rm st,max} \approx 3.78$
- Method: SRGSPH [**KK**, Inutsuka, Seno revised]
- Parallelization: FDPS [Iwasawa+ 2016; Namekata+ 2018]

Result





In Steady-State, **M = 1** at the **Opening**



- **Initially Ejected Gas:**
  Accelerates to the 1D Shock Tube Limit
- **Subsequent Gas:**
  Accelerates to the Bernoulli Limit

Summary

Relativistic SPH

- We propose a new SPH method based on **convolution-integral discretization** and the use of a **Riemann solver**.

$$\begin{cases} N_i = \sum_j \nu_i W(\boldsymbol{x}_i - \boldsymbol{x}_j, h(\boldsymbol{x}_i)) \\ \nu_i \dot{\boldsymbol{S}}_i = -\sum_j P^*_{ij} V^2_{ij} \boldsymbol{F}_{ij} \\ \nu_i \dot{e}_i = -\sum_j P^*_{ij} \boldsymbol{v}^*_{ij} \cdot \boldsymbol{F}_{ij} \end{cases}$$

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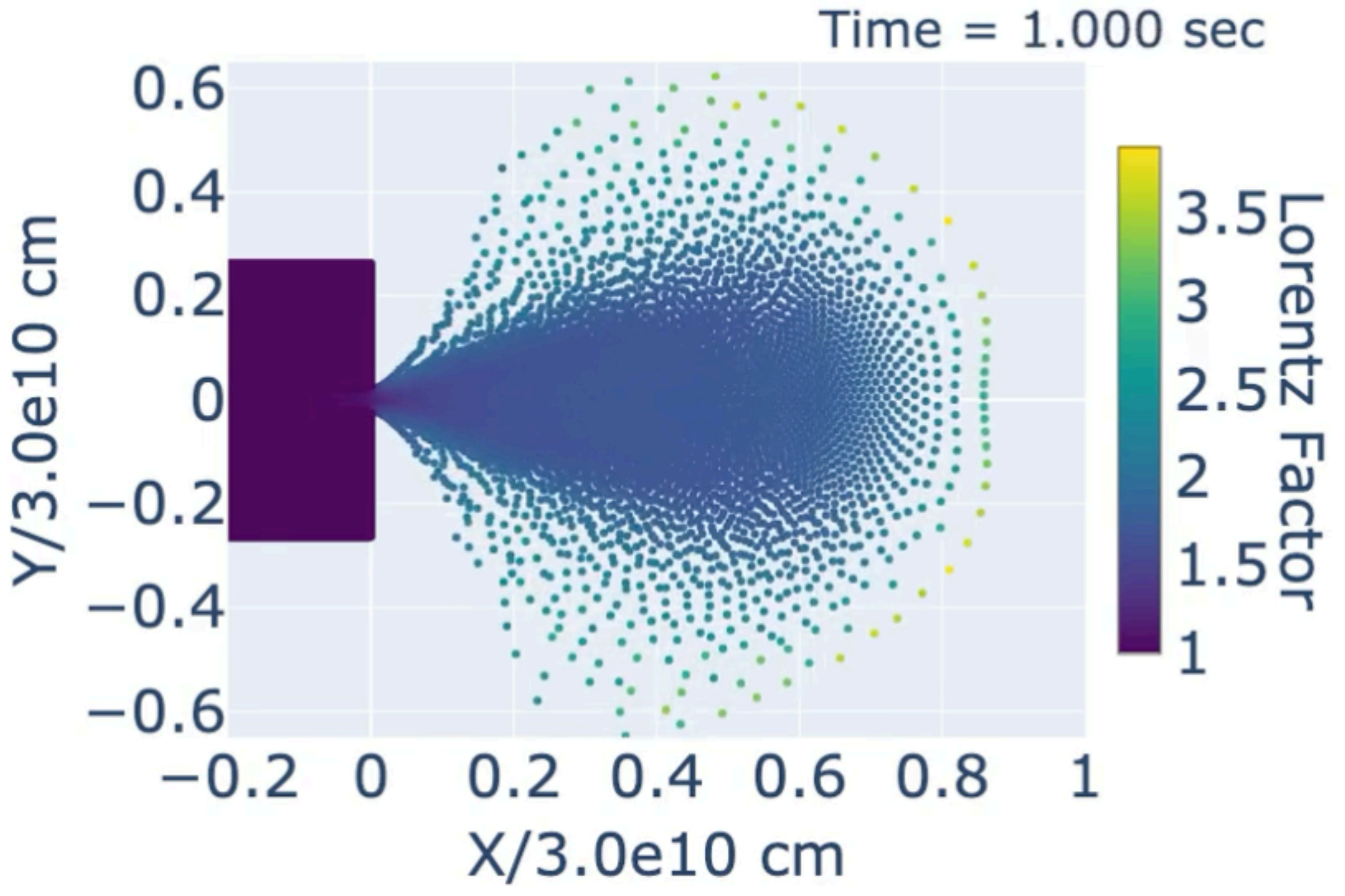
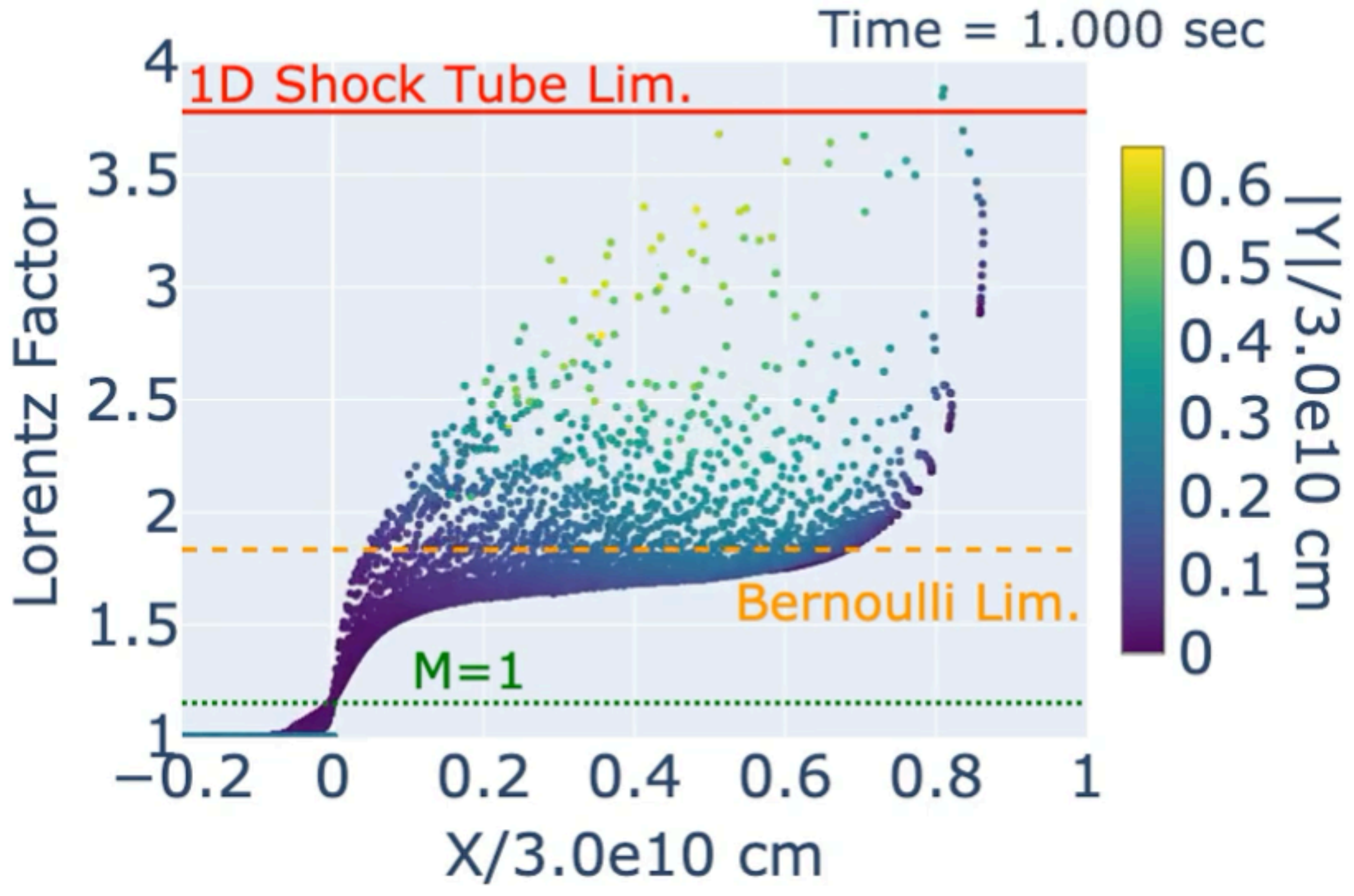
Riemann Solver

Relativistic Jet

- High-Temperature, High-Density Gas Like a Cocoon **Behaves as a Jet**!
- Jets into **Vacuum** Can Be Understood Through **Shock Tube Acceleration** and **Steady-State Acceleration**.





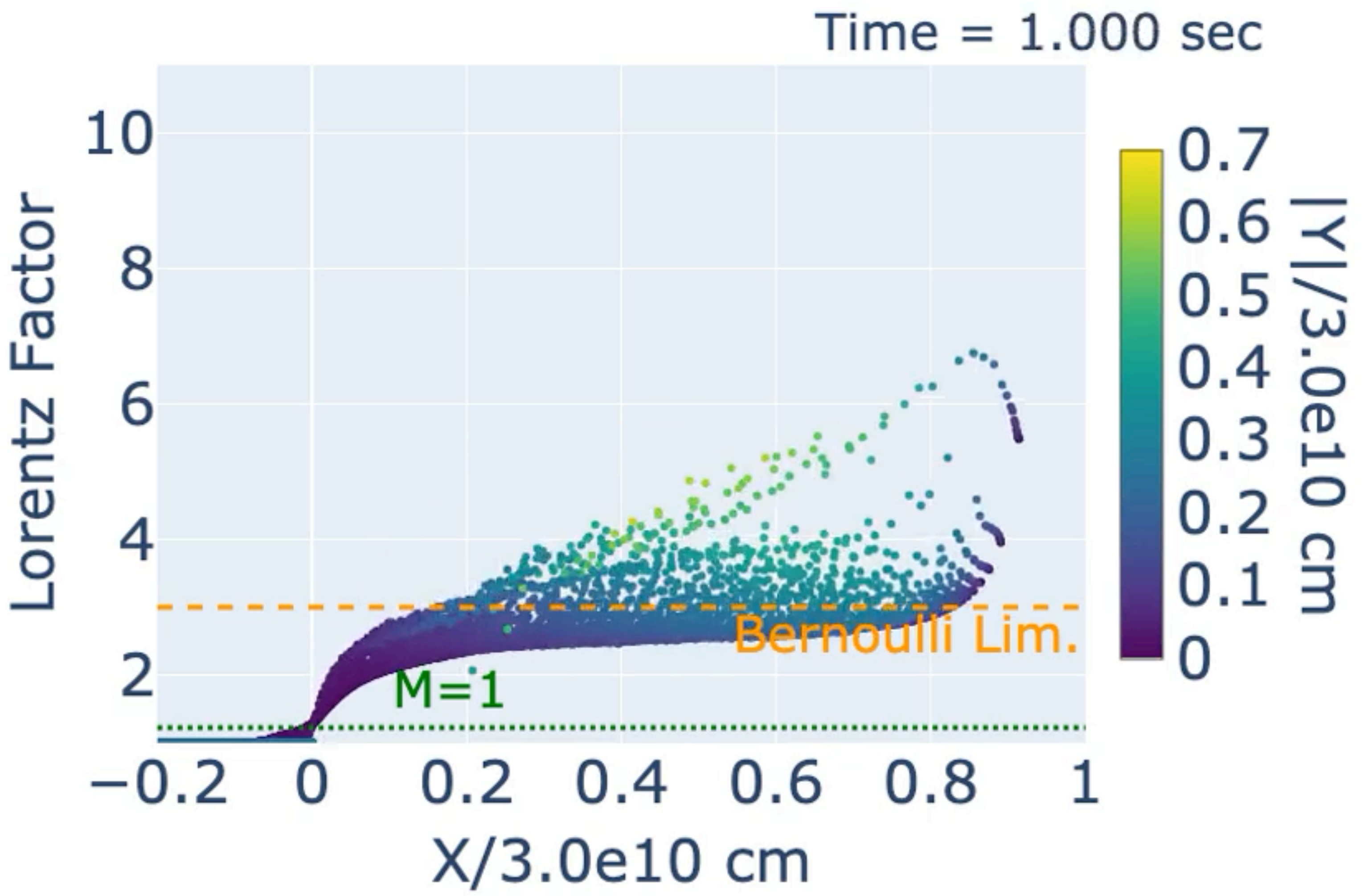
Appendix

Future Prospects | Dependence on Initial Specific Enthalpy

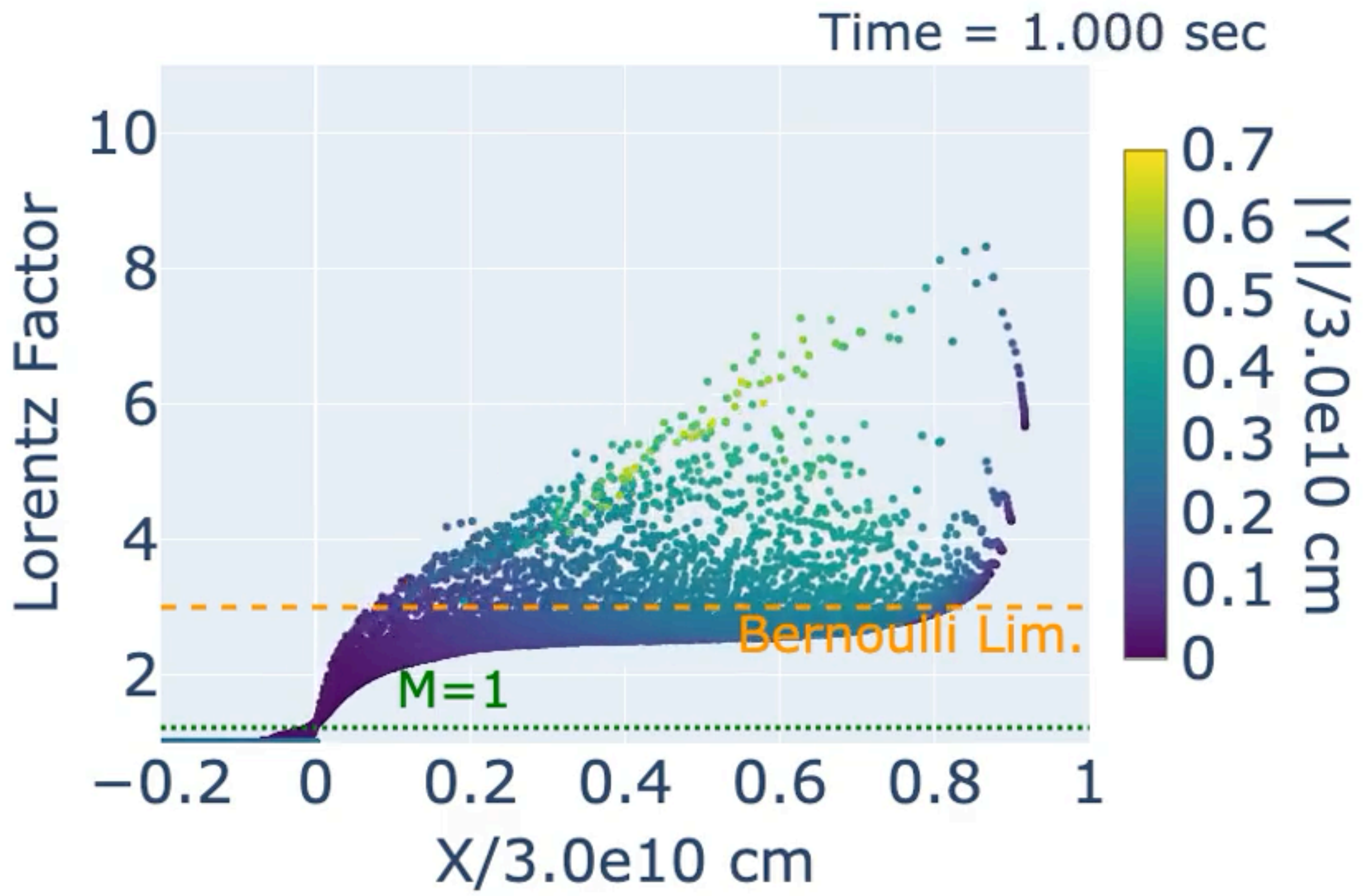
- **Compute for $H_{\text{ini}} = 3$ and compare with the previous case ($H_{\text{ini}} = 11/6$) at a higher temperature.**

In this case, 1D Shock Tube Lim. = 26.5

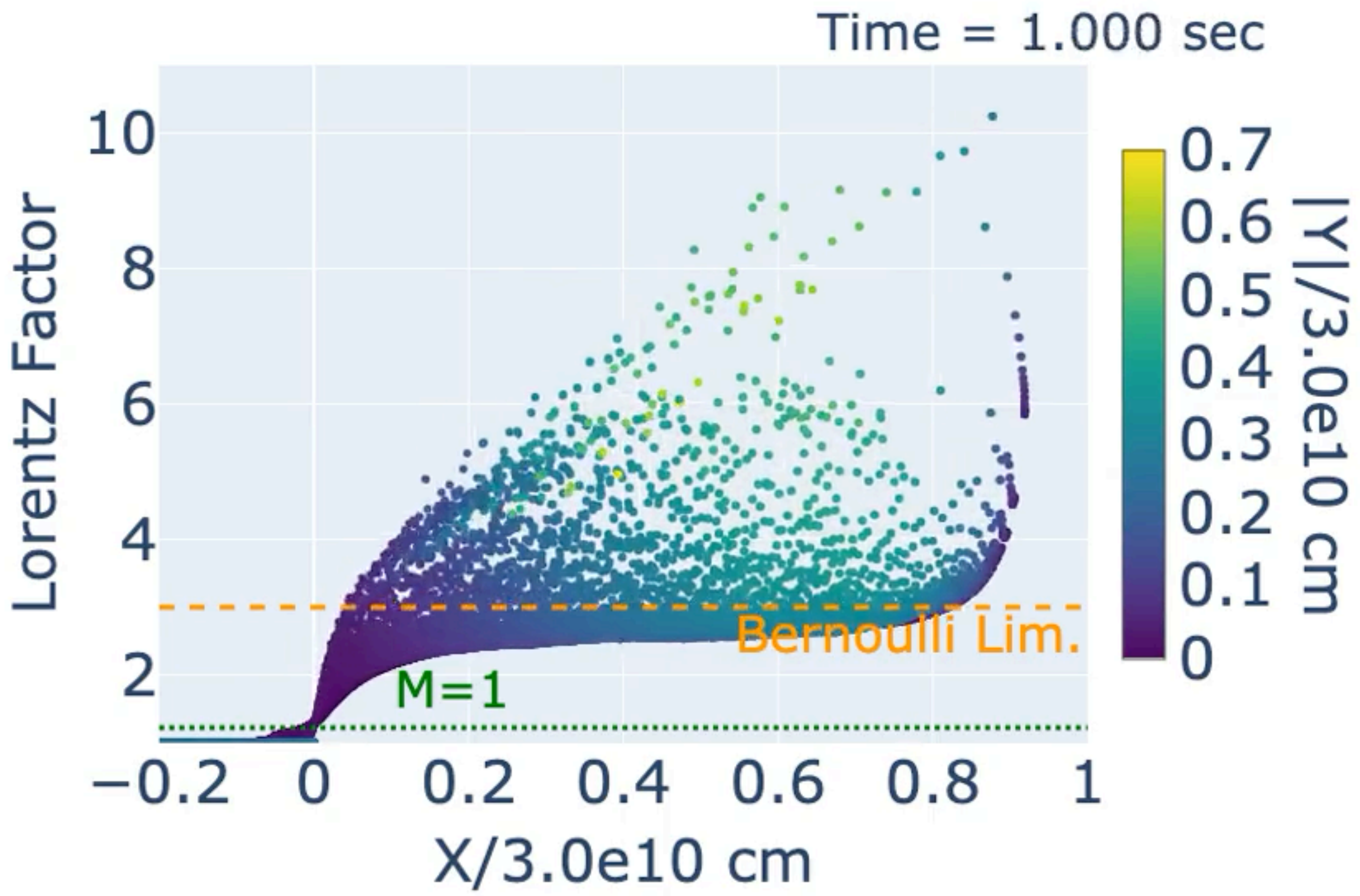
~100,000 Particles



~200,000 Particles



~600,000 Particles



Resolution Low → Hight

Future Plans: Perform Simulations with Significantly Higher Resolution

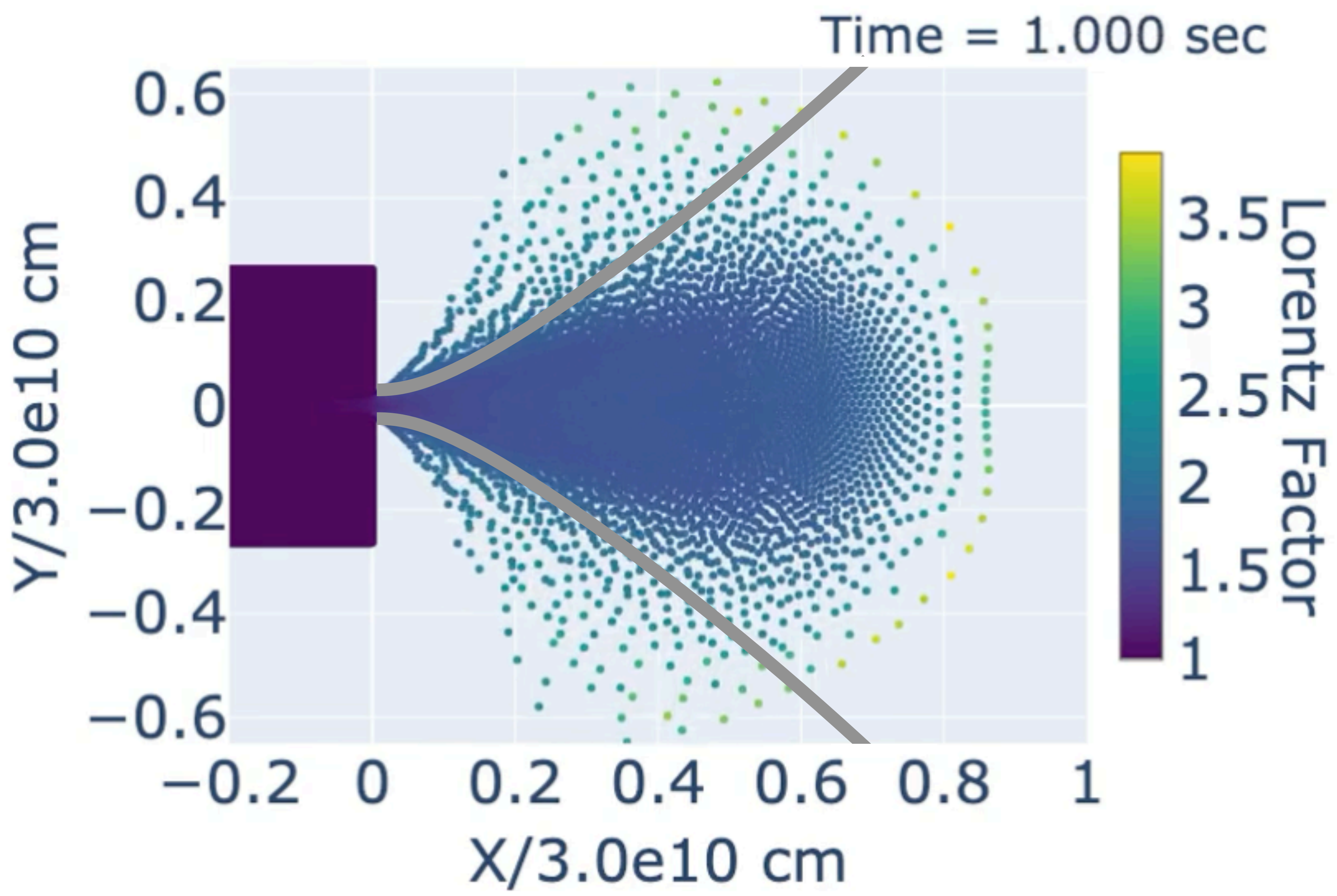
By the way…

γ = 10 → 99.5% of the Speed of Light

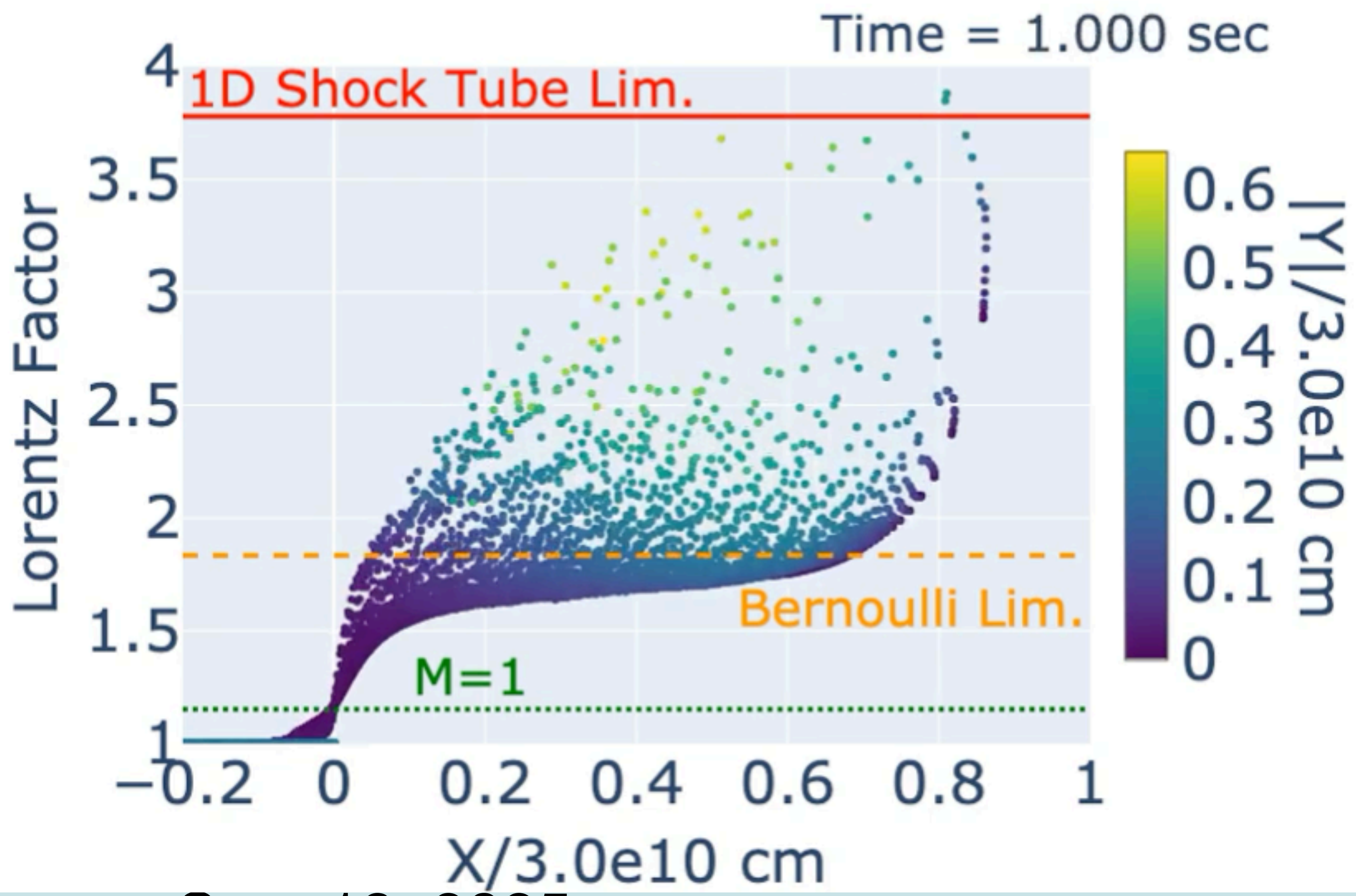
γ = 26.5 → 99.93% of the Speed of Light

2D vs 3D

2D

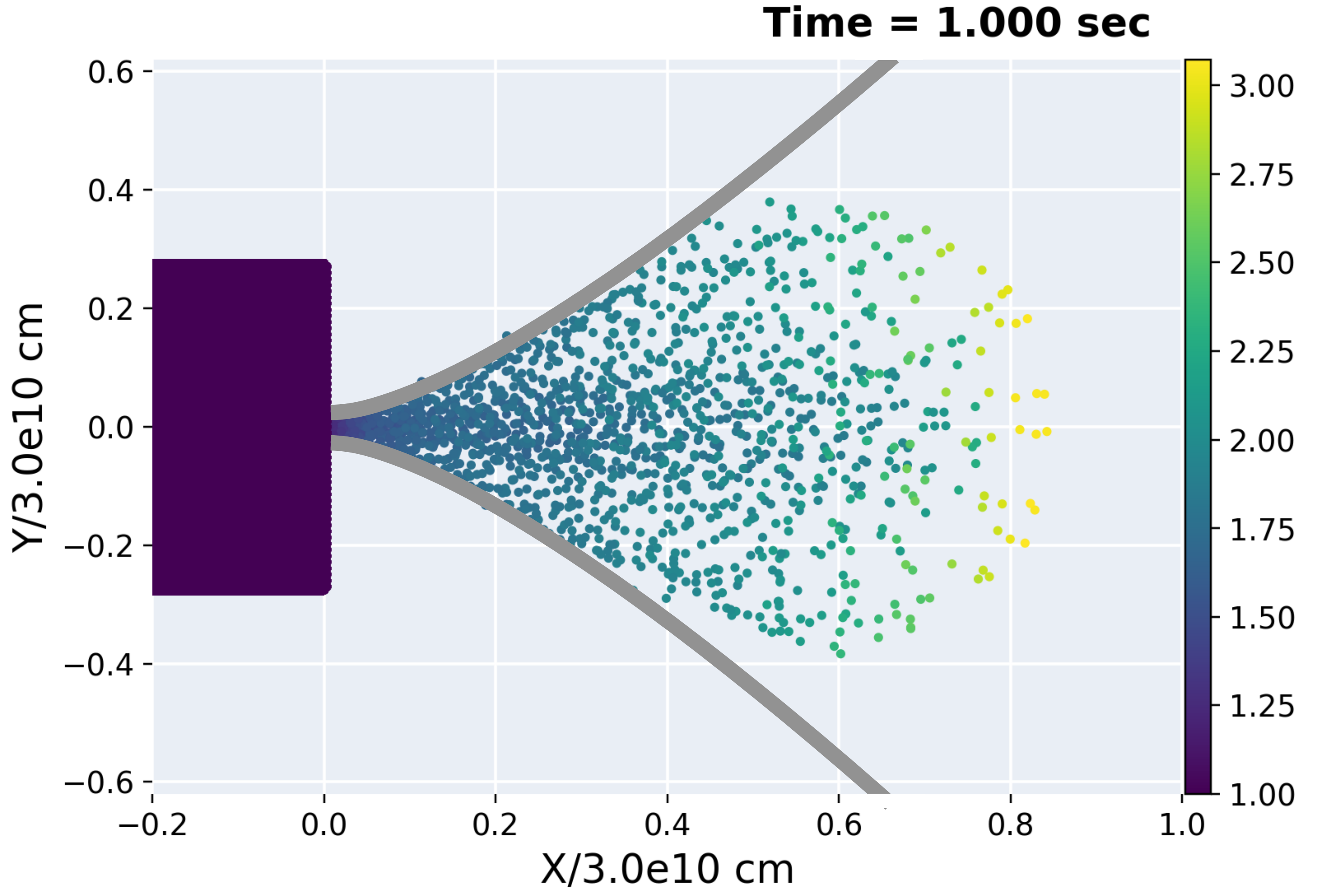


Time = 1.000 sec
Y/3.0e10 cm
X/3.0e10 cm
Lorentz Factor
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.2
0
0.2
0.4
0.6
0.8
1
3.5
3
2.5
2
1.5
1

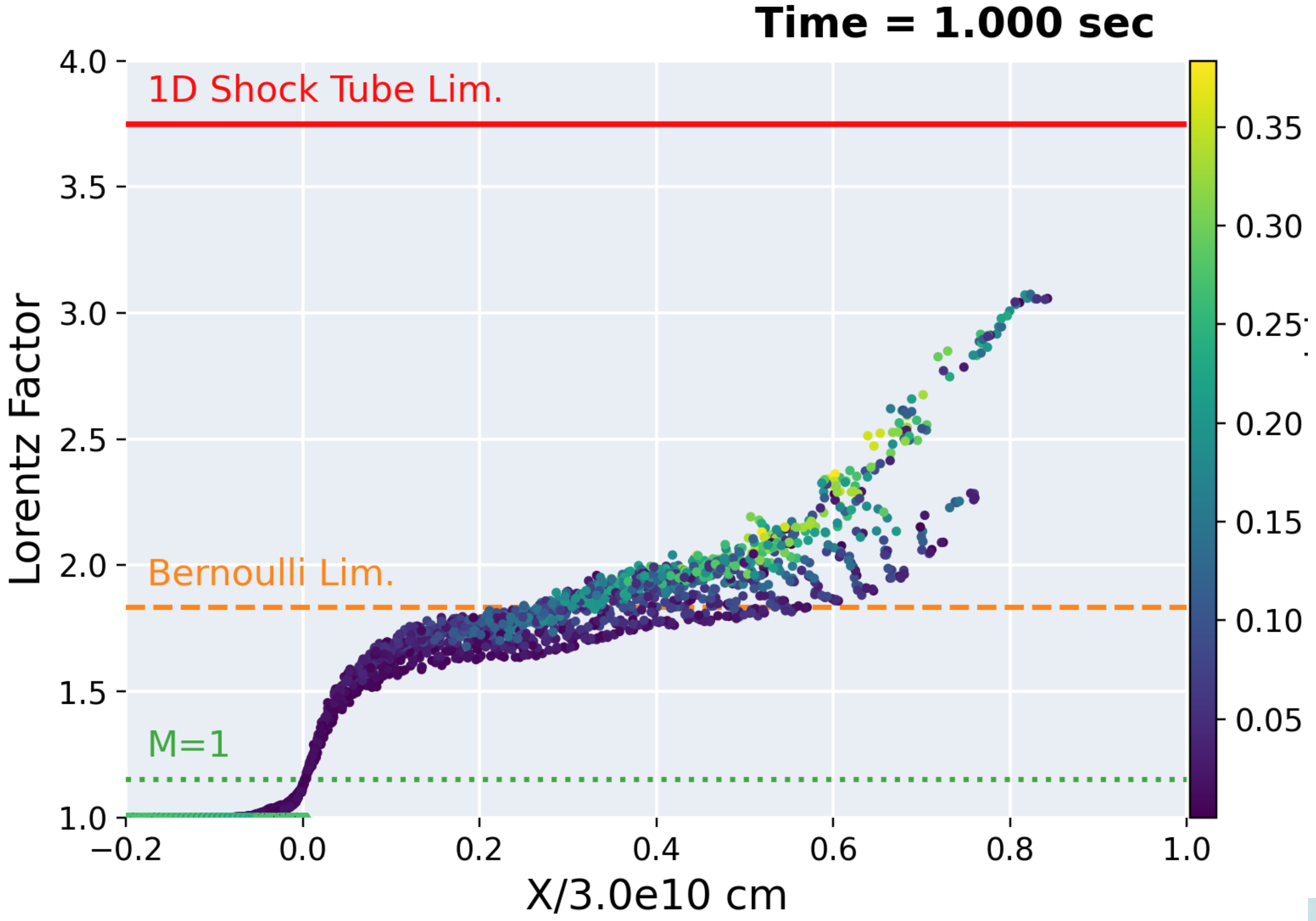


Time = 1.000 sec
1D Shock Tube Lim.
Bernoulli Lim.
M=1
Lorentz Factor
X/3.0e10 cm
|Y|/3.0e10 cm
4
3.5
3
2.5
2
1.5
1
−0.2
0
0.2
0.4
0.6
0.8
1
0.6
0.5
0.4
0.3
0.2
0.1
0

3D



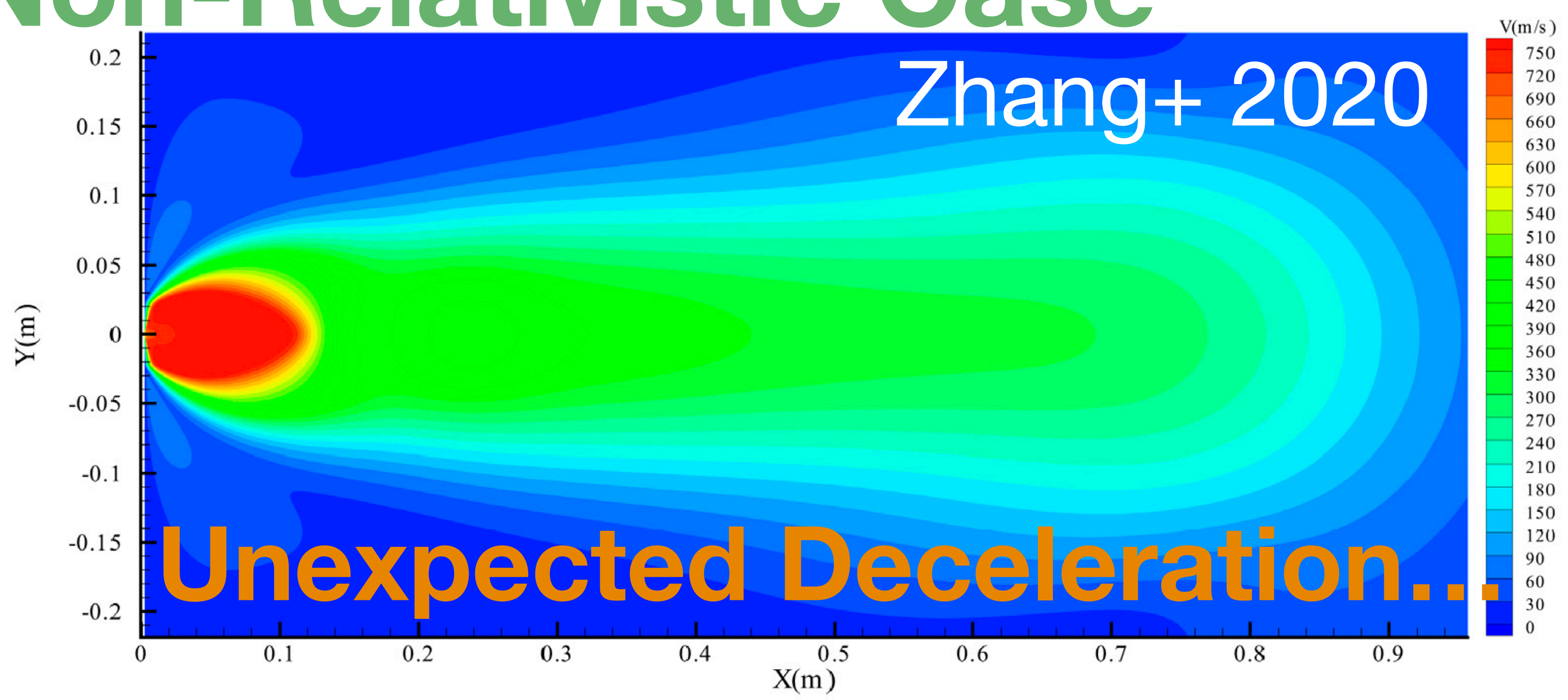
Time = 1.000 sec
Y/3.0e10 cm
X/3.0e10 cm
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
−0.2
0.0
0.2
0.4
0.6
0.8
1.0
3.00
2.75
2.50
2.25
2.00
1.75
1.50
1.25
1.00



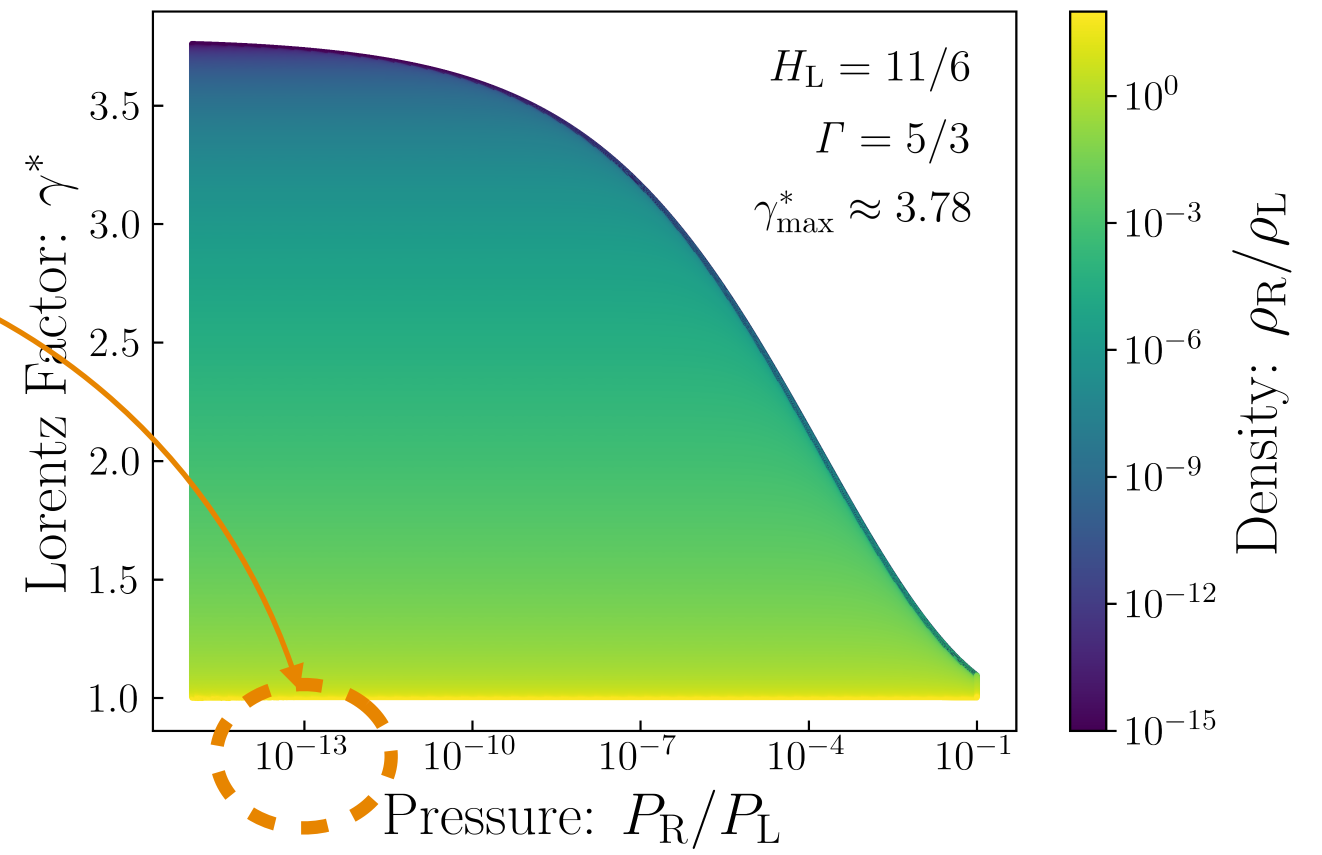
Time = 1.000 sec
1D Shock Tube Lim.
Bernoulli Lim.
M=1
Lorentz Factor
X/3.0e10 cm
4.0
3.5
3.0
2.5
2.0
1.5
1.0
−0.2
0.0
0.2
0.4
0.6
0.8
1.0
0.35
0.30
0.25
0.20
0.15
0.10
0.05

Finite Volume Methods and the Difficulty of Vacuum Expansion

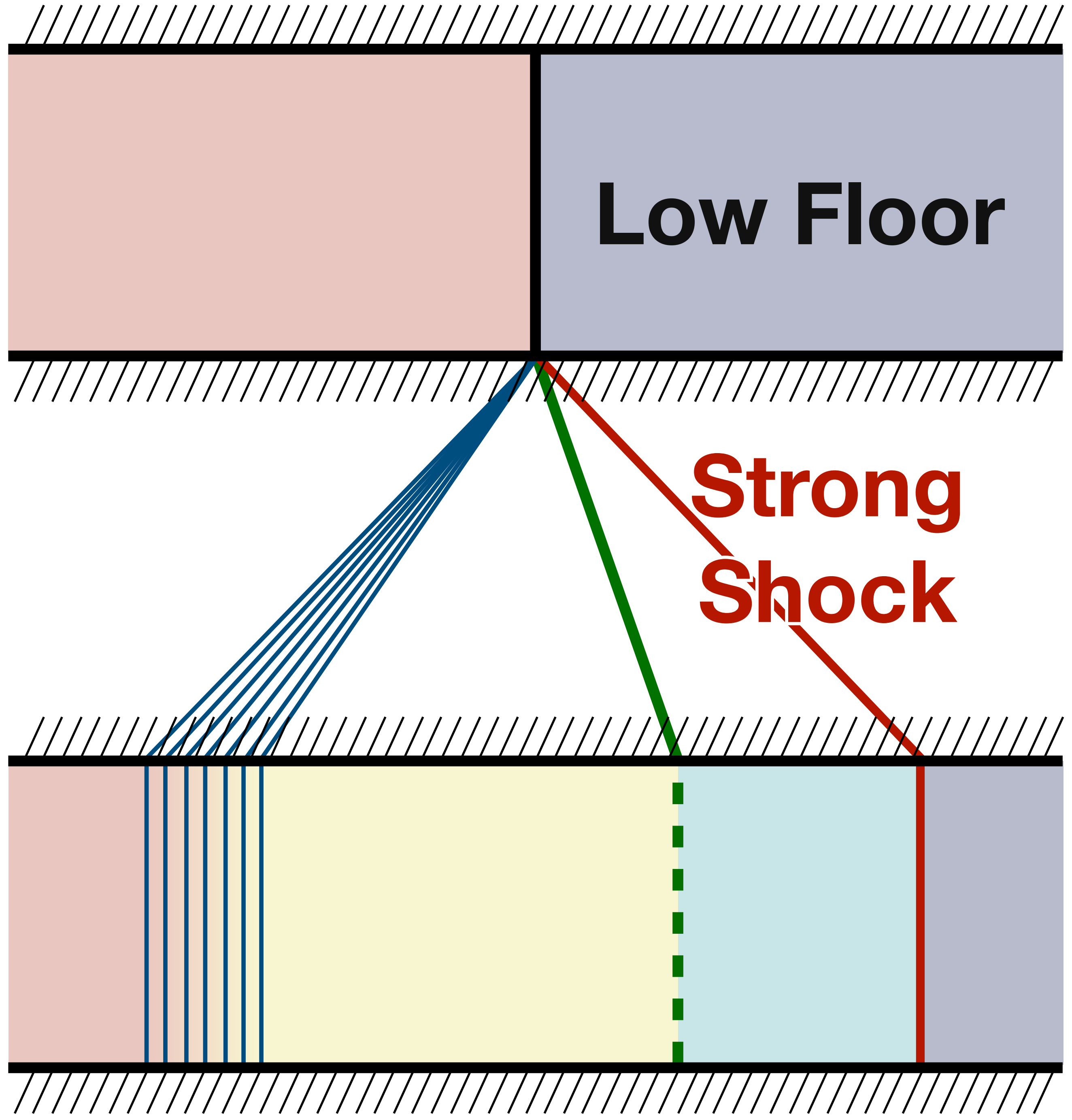
What prevents grid-based methods from calculating accurate acceleration?



- **Extremely low floor values** below 10^{-13} are necessary for accurate acceleration calculations.



- **Low floor values** can lead to **strong shocks**, which in turn cause **numerical instability**.

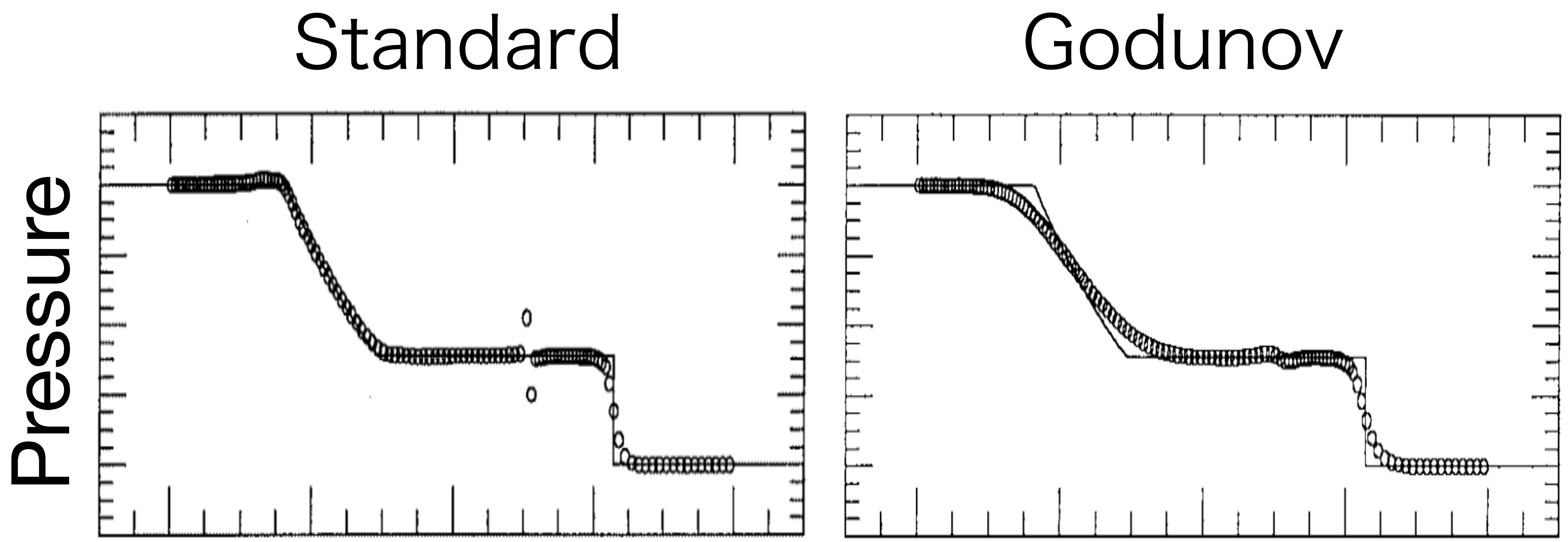


The SPH method can **easily describe vacuum regions by simply not placing particles there**, so such difficulties do not arise.

Comparison with Standard SPH

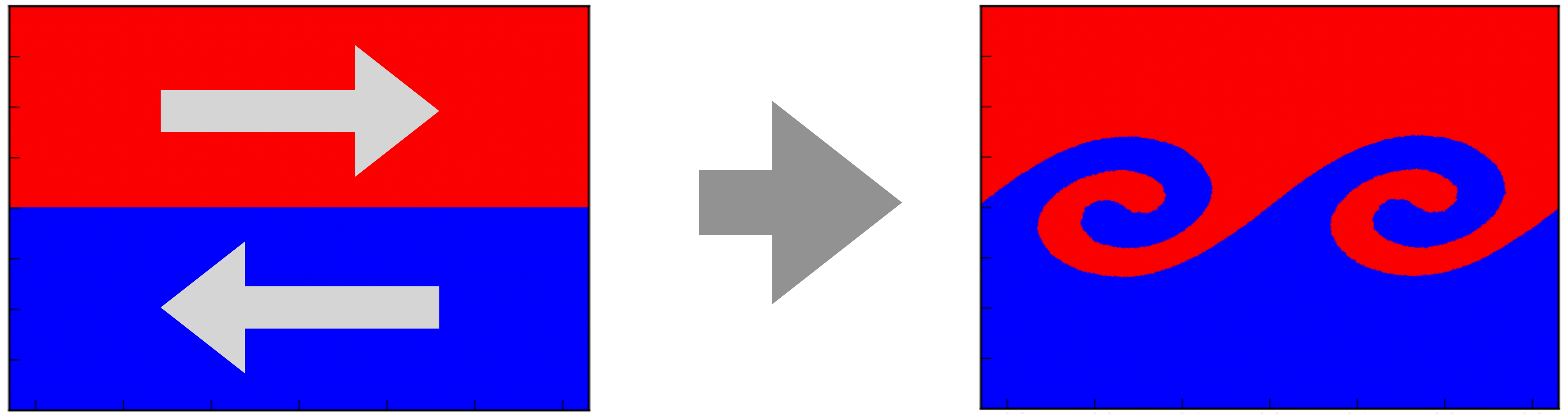
- When multiple shock waves occur, our Godunov-Type method automatically applies an appropriate amount of viscosity.
- Instabilities can be described without the artificial thermal conduction.

As a future prospect, these advantages are expected to become more significant under more complex computational conditions.



Inutsuka (2002)

The Godunov method yields smaller pressure errors at the contact discontinuity than the standard method.



Unlike standard methods, our method does not introduce unphysical thermal conduction when the Kelvin–Helmholtz instability develops.